The equational theory of the natural join and of the inner union is decidable[1]

Luigi Santocanale
LIF, Aix-Marseille Université

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[1] Preprint available on HAL:
https://hal.archives-ouvertes.fr/hal-01625134/

Plan

Algebra from databases

Some undecidable theories

The structure of relational lattices

Decidability of the equational theory of the relational lattices

Plan

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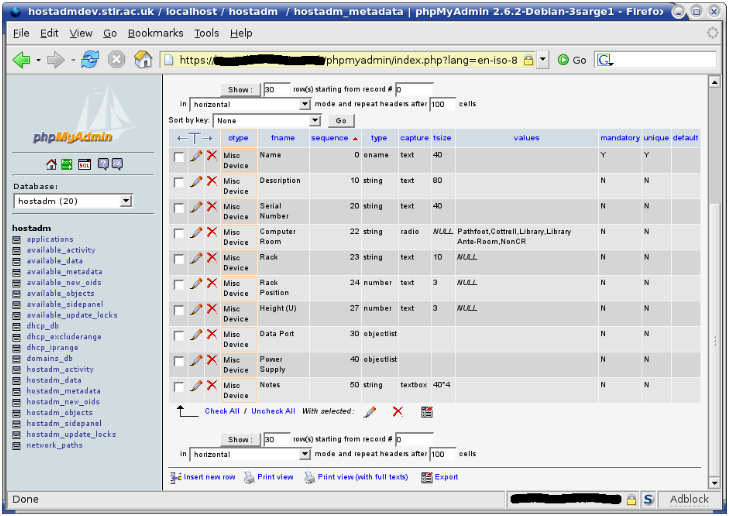
Databases, tables, sqls . . .



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Misc Device	Name	0	oname	text	40		Y	Y	
Misc Device	Description	10	string	text	80		N	N	
Misc Device	Serial Number	20	string	text	40		N	N	
Misc Device	Computer Room	22	string	radio	NULL	Pathfoot,Cottrell,Library,Library Ante-Room,NonCR	N	N	
Misc Device	Rack	23	string	text	10	NULL	N	N	
Misc Device	Rack Position	24	number	text	3	NULL	N	N	
Misc Device	Height (U)	27	number	text	3	NULL	N	N	
Misc Device	Data Port	30	objectlist				N	N	
Misc Device	Power Supply	40	objectlist				N	N	
Misc Device	Notes	50	string	textbox	40*4		N	N	

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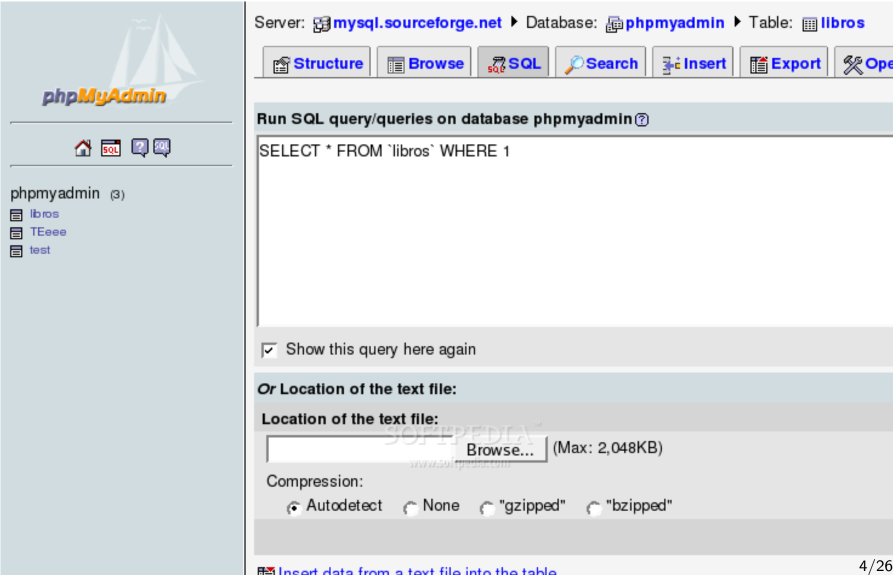

Operations on tables: the natural join

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\bowtie$

Item	Description
33	Book
33	Livre
21	Machine

$=$

Name	Surname	Item	Description
Luigi	Santocanale	33	Book
Luigi	Santocanale	33	Livre
Alan	Turing	21	Machine

Operations on tables: the inner union

Name	Surname	Item
Luigi	Santocanale	33
Alan	Turing	21

$\cup$

Name	Surname	Sport
Diego	Maradona	Football
Usain	Bolt	Athletics

$=$

Name	Surname
Luigi	Santocanale
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Diego	Maradona
Usain	Bolt

Lattices from databases

Proposition. [Spight & Tropashko, 2006] The set of tables, whose columns are indexed by a subset of A and values are from a set D, is a lattice, with natural join as meet and inner union as join.

$\mathrm{R}(D, A)$ shall denote the lattice whose elements are tables, with columns indexed a subset of A and cells' values are from a set D.

A project (Tropashko). Rebuild Codd's relational algebra out of lattice theoretic building blocks.

Plan

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A family of undecidable theories and problems

Theorem (Maddux)

The equational theory of 3-dimensional diagonal free cylindric algebras is undecidable.

Theorem (Hirsch and Hodkinson)

It is not decidable whether a finite simple relation algebra embeds into a concrete one (a powerset of a binary product).

Theorem (Hirsch, Hodkinson and Kurucz)

It is not decidable whether a finite mutimodal frame has a surjective p-morphism from a universal product frame.

n-dimensional diagonal free cylindric algebras, aka the multidimensional modal logic $\mathbf{S5}^n$

- n-dimensional cylindric algebras:
  algebraic modelling of first order logic with no more than n variables.
  Diagonal free: no equality.
- n-multimodal logic **S5**: we have n modal operators $\langle i \rangle$, $i = 1, \ldots, n$, each one is **S5**.
- $\mathbf{S5}^n$ is the n-multimodal logic determined by the universal product frames. These are product sets

  $$X_1 \times \ldots \times X_n$$

  with accessibility given by:

  $$(x_1, \ldots, x_n) R_i (y_1, \ldots, y_n) \text{ iff } x_j = y_j, \text{ for all } j \neq i.$$

- For $n \geq 3$, $\mathbf{S5}^n$ has the finite model property, it is recursively enumerable, yet it is not decidable.

Quasiequations, equations

- A *quasiequation* (definite Horn clause) is the universal closure of a formula of the form

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies s_0 = t_0 \,,$$

with s_i, t_i, $i = 0, \ldots, n$, terms build over a fixed signature.

- The *quasiequational theory* of a class $\mathcal{K}$: the set of quasiequations holding in all elements of $\mathcal{K}$.

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- The *quasiequational theory* of a class $\mathcal{K}$: the set of quasiequations holding in all elements of $\mathcal{K}$.

- An *equation* is a quasiequation as above with $n = 0$.

- The *equational theory* of a class $\mathcal{K}$: the set of equations holding in all elements of $\mathcal{K}$.
  See the standard Birkhoff's theorem.

Undecidable quasiequational theories of relational lattices

Theorem (Litak, Mikulás and Hidders, 2015)

The set of quasiequations in the signature $(\wedge, \vee, H)$ *that are valid on relational lattices is undecidable.*

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This was refined to:

Theorem (Santocanale, RAMICS 2017)

The set of quasiequations in the signature $(\wedge, \vee)$ *that are valid on relational lattices is undecidable.*

where we actually proved a stronger result:

Theorem (Santocanale 2017)

It is undecidable whether a finite subdirectly irreducible lattice embeds into some $\mathrm{R}(D, A)$.

Plan

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The relational lattices $\mathsf{R}(D, A)$

A a set of attributes, D a set of values.

An element of $\mathsf{R}(D, A)$:

- a pair (α, Y) with $\alpha \subseteq A$ and $Y \subseteq D^{\alpha}$.

We have

$$(\alpha_1, Y_1) \leq (\alpha_2, Y_2) \text{ iff } \alpha_2 \subseteq \alpha_1 \text{ and } Y_1{\upharpoonright\!\upharpoonright}_{\alpha_2} \subseteq Y_2 \,.$$

NB :

- $\upharpoonright\!\upharpoonright$ is restriction:

$$Y{\upharpoonright\!\upharpoonright}_{\alpha} = \{\, f_{\upharpoonright_{\alpha}} \mid f \in Y \,\} \,.$$

Meet and join

$$
\begin{aligned}
(\alpha_1, Y_1) \wedge (\alpha_2, Y_2) &:= (\alpha_1 \cup \alpha_2, Y) \\
\text{where } Y &= \{ f \mid f_{\restriction \alpha_i} \in Y_i, i = 1, 2 \} \\
&= i_{\alpha_1 \cup \alpha_2}(Y_1) \cap i_{\alpha_1 \cup \alpha_2}(Y_2)\,,
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_1, Y_1) \vee (\alpha_2, Y_2) &:= (\alpha_1 \cap \alpha_2, Y) \\
\text{where } Y &= \{ f \mid \exists i \in \{1, 2\}, \exists g \in Y_i \text{ s.t. } g_{\restriction \alpha_1 \cap \alpha_2} = f \} \\
&= Y_1 {\Vert\!\restriction}_{\alpha_1 \cap \alpha_2} \cup Y_2 {\Vert\!\restriction}_{\alpha_1 \cap \alpha_2}\ .
\end{aligned}
$$

NB :

- i is cylindrification:

$$
i_\alpha(Y) = \{ f \mid f_{\restriction \alpha} \in Y \}\,.
$$

Representation via closure operators

The Hamming/Priess_Crampe-Ribenboim ultrametric distance on D^A:

$$\delta(f, g) := \{ x \in A \mid f(x) \neq g(x) \} .$$

NB: this distance takes values in the join-semilattice $(P(A), \emptyset, \cup)$.

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A subset Z of $A + D^A$ is *closed* if

$$\begin{pmatrix} \delta(f, g) \subseteq A \cap Z \\ g \in D^A \cap Z \end{pmatrix} \text{ implies } f \in Z .$$

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Proposition. [Litak, Mikulás and Hidders 2015] $\mathrm{R}(D,A)$ is isomorphic to the lattice of closed subsets of $A + D^A$.

Lattices from generalized ultrametric spaces

In a similar way, we can construct a lattice from any generalized ultrametric space (X, δ) over some $P(A)$.

A subset $Z \in P(A + X)$ is *closed* if

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A pair $(\alpha, Y) \in P(A) \times P(Y)$ is *closed* if

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Thus we put

$$\mathsf{L}(X,\delta) := \{\, (\alpha, Y) \mid \langle\alpha\rangle Y \subseteq Y \,\}\,,$$

where

$$\langle\alpha\rangle Y = \{\, f \in X \mid \exists g \in Y \,\text{s.t.}\, \delta(f,g) \subseteq \alpha \,\}\,.$$

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Clearly $\mathsf{R}(D, A) = \mathsf{L}(D^A, \delta)$.

Universal product spaces as injective generalized ultrametric spaces

Lemma

TFAE :

- (X, δ) *is injective in the category of generalized ultrametric spaces over* $P(A)$*,*
- (X, δ) *is, up to isomorphism, a universal product space:*

$$X = \prod_{a \in A} X_a \,, \qquad \delta(x, y) = \{\, a \in A \mid x_a \neq y_a \,\} \,.$$

Remark : intuitively, injective means complete.

Relational lattices as modal logic

We can interpret the theory of the lattices $\mathsf{L}(X, \delta)$ in a sort of multidimensional $\mathbf{S5}^n$ modal logic. Modal operators are indexed by subsets of A:

$$\langle \alpha \rangle Y := \{ f \in D^A \mid \exists g \in Y \text{ s.t. } \delta(f, g) \subseteq \alpha \} .$$

If (X, δ) is injective, then we have the following logical equivalence:

$$\langle \alpha_1 \cup \alpha_2 \rangle Y = \langle \alpha_1 \rangle \langle \alpha_2 \rangle Y .$$

Meet is conjunction, where the join is:

$$\begin{aligned}(\alpha_1, Y_1) \vee (\alpha_2, Y_2) &= (\alpha_1 \cup \alpha_2, \langle \alpha_1 \cup \alpha_2 \rangle (Y_1 \cup Y_2)) \\ &= (\alpha_1 \cup \alpha_2, \langle \alpha_1 \cup \alpha_2 \rangle Y_1 \cup \langle \alpha_1 \cup \alpha_2 \rangle Y_2) \\ &= (\alpha_1 \cup \alpha_2, \langle \alpha_2 \rangle \langle \alpha_1 \rangle Y_1 \cup \langle \alpha_1 \rangle \langle \alpha_2 \rangle Y_2) \\ &= (\alpha_1 \cup \alpha_2, \langle \alpha_2 \rangle Y_1 \cup \langle \alpha_1 \rangle Y_2) .\end{aligned}$$

Plan

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- The method is reminiscent of Gabbay's selective filtration in modal logic.
- Thanks to TICAMORE: I would have not found this, if had not wondered about semantics vs syntactic methods for decidability in modal logic.

The tableau of a failure

Suppose $t \leq s$ is not vaild in all the $\mathsf{R}(D, A)$s.

- For some A, D and $v : \mathsf{X} \to \mathsf{R}(D, A)$, $[\![t]\!]_v \not\subseteq [\![s]\!]_v$.
- We can suppose that $f \in [\![t]\!]_v \setminus [\![s]\!]_v$ for some $f \in D^A$.

Lemma (preservation of failures)

There is a finite subset $T(f, t) \subseteq D^A$ such that, if $T(f, t) \subseteq T \subseteq D^A$, then

$$\mathsf{L}(T, \delta) \not\models t \leq s\,.$$

- Above, (T, δ) is the subspace of (D^A, δ) induced by T.

Failures in a finite lattice

- The lattice $\mathsf{L}(T, \delta)$ might still be infinite, even if T is finite.
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Let T be finite.

- If $\mathcal{B}_T$ is the Boolean algebra generated by

$$\{\, \delta(g, h) \mid g, h \in T \,\} \cup \{\, A \cap v(x) \mid x \in Vars(t, s) \,\},$$

  then $\mathcal{B}_T \simeq P(B_T)$ for some finite subset B_T.
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Lemma (preservation of failures in the finite)

There is a finite subset $T(f,t) \subseteq D^A$ such that, if $T(f,t) \subseteq T \subseteq D^A$ and T is finite, then

$$\mathsf{L}(T, \delta_{B_T}) \not\models t \leq s\,.$$

Failures in a universal product frame

- The $\mathsf{L}(T, \delta_{B_T})$ is finite lattice.
- Yet, it does not need to be in the variety of the $\mathsf{R}(D, A)$s.

Lemma

For each finite T there is a finite $G(T) \subseteq D^A$ such that

- $T \subseteq G(T)$,
- $\mathcal{B}_T = \mathcal{B}_{G(T)}$,
- $(G(T), \delta_{B_{G(T)}})$ *is injective relative to* $P(B_T)$.

Corollary. Let $T_0 := T(f, t)$. Then the lattice $L(G(T_0), \delta_{B_{T_0}})$ is finite and

$$\mathsf{L}(G(T_0), \delta_{B_{T_0}}) \not\models t \leq s\,.$$

Summing up

- If $(G(T_0), \delta_{B_{T_0}})$ is injective then (up to isomorphism)

$$G(T_0) = \prod_{b \in B_{T_0}} X_b \,.$$

- Taking b_0 so X_{b_0} is of maximal cardinality, we can embed $G(T_0)$ into ${X_{b_0}}^{B_{T_0}}$:

$$\prod_{b \in B_{T_0}} X_b \subseteq \prod_{b \in B_{T_0}} X_{b_0} = {X_{b_0}}^{B_{T_0}} \,.$$

- Using injectivity, functoriality, and a bit of lattice theoretic tricks, we can show $\mathrm{L}(G(T_0), \delta_{B_{T_0}})$ is a homomorphic image of a sublattice of $\mathrm{R}(X_{b_0}, B_{T_0})$.
- Then

$$\mathrm{R}(X_{b_0}, B_{T_0}) \not\models t \leq s$$

otherwise we would have $\mathrm{L}(G(T_0), \delta_{B_{T_0}}) \models t \leq s$.

TICAMORE challenges

- complexity issues ;
- axiomatizations and automated tools ;
- completeness ;
- from labeled calculi to pure equational axioms ?