Mutual Translations between Display and Labelled Proofs for Tense Logics

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Joint work with
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1. Background & Questions
   - Background
   - Questions
2. Display & Labelled Calculi
   - Logic of Interest: Tense Logic
   - Display Calculi and Display Sequents
   - Labelled Calculi and Labelled Sequents
3. Current Results
4. Applications & Future Work

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The **Sequent**ial Landscape:

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$A, B \vdash C, D, E$

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$$Rxy, Rxz, x : A, y : B, y : C$$

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$$w, [u, A, [v, B, C]] \owedge w, [u, D, [v, B, C]]$$

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$$w, [u, A, [v, B, C]] \owedge w, [u, D, [v, B, C]]$$

Et cetera . . .

Some Questions:

- How do "sequent languages" compare to each other?
- How does the sequent language shape the space of proofs in the corresponding calculus?
- Internal and External Calculi:
    - What is a satisfactory formal definition of each?
    - What desiderata determine internality or externality?
- What can we learn from the stepwise translation of proofs between calculi built from different languages?

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What is tense logic?

- A logic for reasoning about logical notions of time.
- Language:

$$A := p|\overline{p}|A \vee A|A \wedge A|\Box A|\blacksquare A|\Diamond A|\blacklozenge A$$

- Interpretations:
  - $\Box A$ is interpreted as "A holds at every point in the future."
  - $\blacksquare A$ is interpreted as "A holds at every point in the past."
  - $\Diamond A$ is interpreted as "A holds at some point in the future."
  - $\blacklozenge A$ is interpreted as "A holds at some point in the past."

Tense Logic (Cont.)

Hilbert Calculus:

Axioms:	Inference Rules:
$A \to (B \to A)$	$\dfrac{A \qquad A \to B}{B}$ (MP)
$A \to (B \to C) \to ((A \to B) \to (A \to C))$	
$(A \to B) \to (\neg B \to \neg A)$	
$\Box(A \to B) \to (\Box A \to \Box B)$	$\dfrac{A}{\Box A}$ ($\Box$)
$\Box A \leftrightarrow \neg\Diamond\neg A$	
$\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B)$	$\dfrac{A}{\blacksquare A}$ ($\blacksquare$)
$\blacksquare A \leftrightarrow \neg\blacklozenge\neg A$	
$A \to \Box\blacklozenge A$	
$A \to \blacksquare\Diamond A$	

Definition

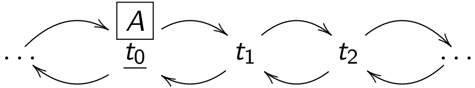
The Minimal Tense Logic Kt

We define the logic Kt to be the smallest set of formulae containing all deductive consequences of the Hilbert calculus above.

Modelling Time

Modelling Time

- Converse Axioms $A \rightarrow \Box\blacklozenge A$ and $A \rightarrow \blacksquare\Diamond A$:



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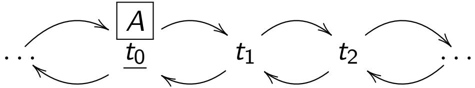
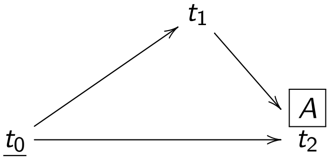


- Transitivity $\Diamond\Diamond A \to \Diamond A$:



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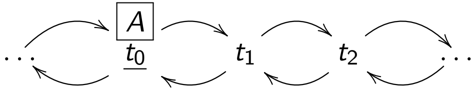

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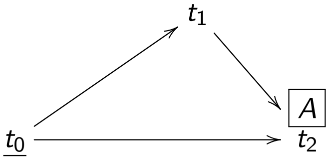
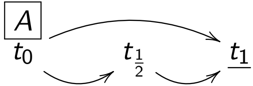


- Density $\blacklozenge A \to \blacklozenge\blacklozenge A$:



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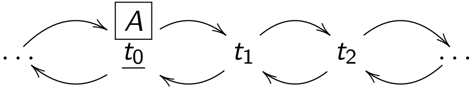

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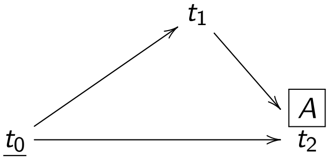

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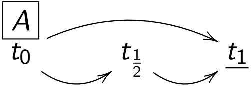

Definition

Scott-Lemmon Axioms: $\blacklozenge^h \Diamond^j p \to \Diamond^i \blacklozenge^k p$ for $h, j, i, k \in \mathbb{N}$

A Display Calculus for Tense Logic

Language:

$$X := A|X, X| \circ \{X\}| \bullet \{X\}$$

where A is a tense logic formula.

Definition

The Display Calculus SKT [Goré *et al.* 2011]:

$$\frac{}{\Gamma, p, \overline{p}}\ (\text{id}) \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ (\vee) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ (\wedge)$$

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ (\text{ctr}) \qquad \frac{\Gamma}{\Gamma, \Delta}\ (\text{wk}) \qquad \frac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta}\ (\text{rf}) \qquad \frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta}\ (\text{rp})$$

$$\frac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A}\ (\blacksquare) \qquad \frac{\Gamma, \circ\{A\}}{\Gamma, \Box A}\ (\Box)$$

$$\frac{\Gamma, \bullet\{\Delta, A\}, \blacklozenge A}{\Gamma, \bullet\{\Delta\}, \blacklozenge A}\ (\blacklozenge) \qquad \frac{\Gamma, \circ\{\Delta, A\}, \Diamond A}{\Gamma, \circ\{\Delta\}, \Diamond A}\ (\Diamond)$$

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$$\frac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A} (\blacksquare) \qquad \frac{\Gamma, \circ\{A\}}{\Gamma, \Box A} (\Box)$$

$$\frac{\Gamma, \bullet\{\Delta, A\}, \blacklozenge A}{\Gamma, \bullet\{\Delta\}, \blacklozenge A} (\blacklozenge) \qquad \frac{\Gamma, \circ\{\Delta, A\}, \Diamond A}{\Gamma, \circ\{\Delta\}, \Diamond A} (\Diamond)$$

Scott-Lemmon Rules (Display)

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}}\,\delta SL \quad \equiv \quad \blacklozenge^h\lozenge^j p \to \lozenge^i\blacklozenge^k p \; + \; (\text{cut})$$

- The δSL structural rule preserves cut-admissibility when added to SKT.
  - Actually, this holds for a much larger class of rules: Primitive Tense Structural Rules [Kracht 1996].
- The structural rule is equivalent to extending with the corresponding axiom.

SKT is Internal with respect to Kt: Why?

- We think of a calculus intuitively as internal if it exists in a language where each sequent "naturally" corresponds (is equivalent) to a formula of the object language.
- Example:

$$A, \bullet\{\circ\{B,\ C\},\ D\},\ \circ\{E\}$$
$$A \vee \blacksquare(\Box(B \vee C) \vee D) \vee \Box E$$

- We can read:
  - Comma , as $\vee$
  - White circle $\circ$ as $\Box$
  - Black circle $\bullet$ as $\blacksquare$
  - The nestings $\{\cdot\}$ as the scope of the corresponding operator.

Visualizing Display Sequents

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$

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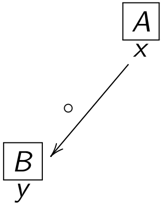

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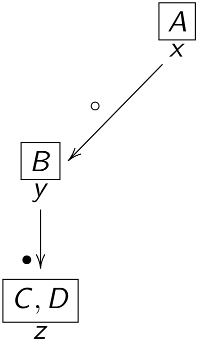

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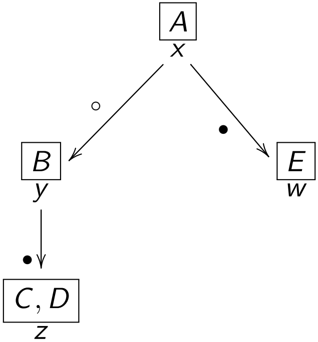

A Labelled Calculus for Tense Logic

Language:

$$X := x : A | X, X | Rxy, X$$

where A is a tense logic formula.

Definition

The labelled sequent calculus G3Kt [Negri 2005]:

$$\frac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma}\ (\mathrm{id})$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma}\ (\vee) \qquad \frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma}\ (\wedge)$$

$$\frac{\mathcal{R}, Ryx, y : A, \Gamma}{\mathcal{R}, x : \blacksquare A, \Gamma}\ (\blacksquare)^{*} \qquad \frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma}\ (\Box)^{*}$$

$$\frac{\mathcal{R}, Ryx, y : A, x : \blacklozenge A, \Gamma}{\mathcal{R}, Ryx, x : \blacklozenge A, \Gamma}\ (\blacklozenge) \qquad \frac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma}\ (\Diamond)$$

Scott-Lemmon Rules (Labelled)

- Obtained as instances (with substitutions done by hand) from the following rule scheme:

$$\frac{\mathcal{R}, R^i vx, R^k ux, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma}\ \lambda SL$$

- Equivalent to both:

$$\text{(i) } R^h wv \wedge R^j wu \rightarrow \exists x (R^i vx \vee R^k ux) \quad \text{(ii) } \blacklozenge^h \Diamond^j p \rightarrow \Diamond^i \blacklozenge^k p$$

- Structural rule is equivalent to extending with axiom.
- Addition of λSL preserves cut-admissibility.

Visualizing Labelled Sequents

$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$

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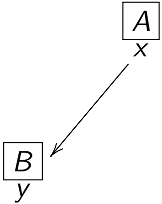

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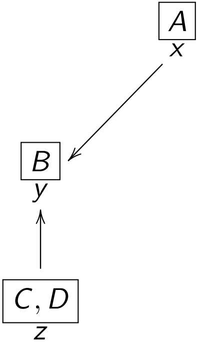

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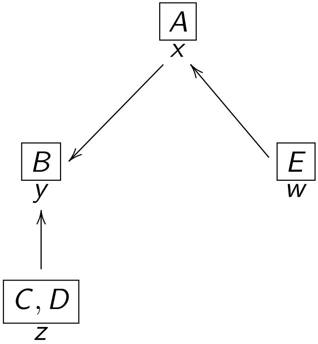

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Embedding Formalisms: Related Work

- Ramanayake "Embedding the hypersequent calculus in the display calculus" 2014

$$\text{Hypersequent} \subseteq \text{Display}$$

- Goré and Ramanayake "Labelled Tree Sequents, Tree Hypersequents and Nested (Deep) Sequents" 2012

$$\text{Nested} \subseteq \text{Labelled}$$

- Greg Restall "Comparing Modal Sequent Systems" 2006

$$\text{Display} \subseteq_{K} \text{Labelled}$$

New Results

- Ciabattoni, Lyon, Ramanayake "From Display to Labelled Proofs for Tense Logics" (to appear in Logical Foundations of Computer Science 2018 Proceedings)

- How does it expand on previous work?
  - For the logic Kt it shows how to translate between display and labelled proofs *of formulae*, *i.e.*

$$\text{Display} =_{\text{Kt}} \text{Labelled}$$

  - For the logic Kt extended with Scott-Lemmon it shows:

$$\text{Display} \subseteq_{\text{Kt+SL}} \text{Labelled}$$

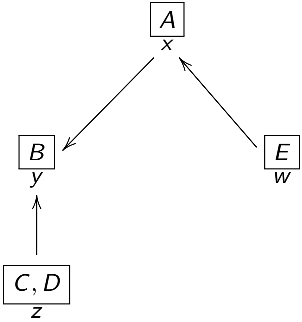
Roadmap of Results

- Introduction: Labelled UT Sequents
- Translating Display Sequents into Labelled UT sequents
- Translating Labelled UT sequents into Display Sequents
- Main Theorems

Which Labelled Sequents are Essentially Display?

- What is a Labelled UT?
  - A labelled UT is a directed graph whose underlying graph (also called a *shadow*) is a tree (connected and acyclic).
  - A labelled UT sequent is a sequent whose graph is a labelled UT.
- Example:

  $Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



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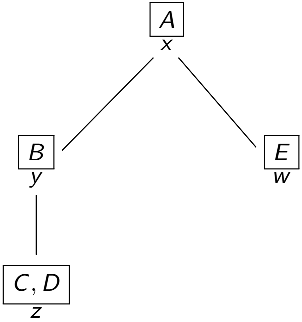
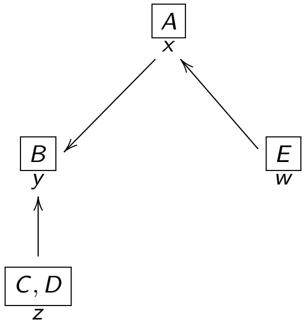


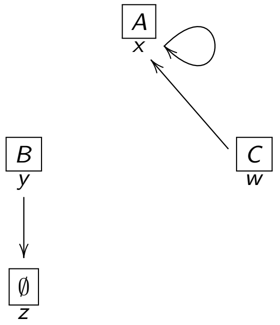
Comparing General Labelled and UT Labelled

- Some sequents naturally have labelled UT graphs, and others do not:

$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$Rxx, Ryz, Rux, x : A, y : B, u : C$



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  - (1) Flip • edges and switch type
  - (2) Remove edge-typing

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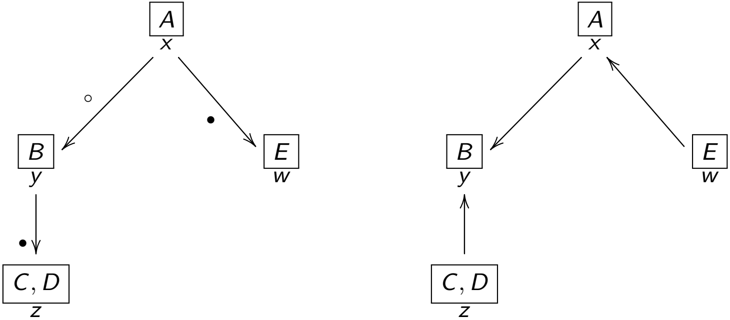

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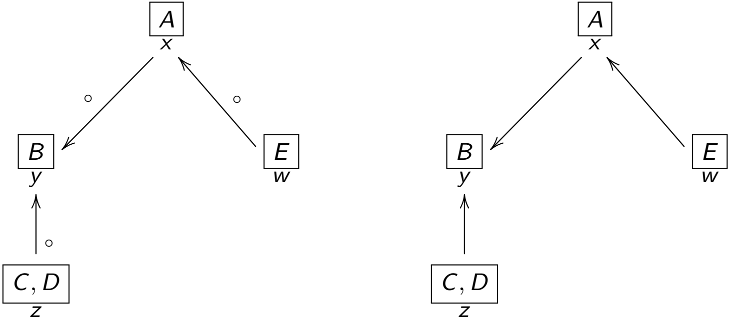

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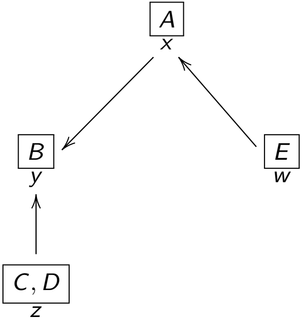

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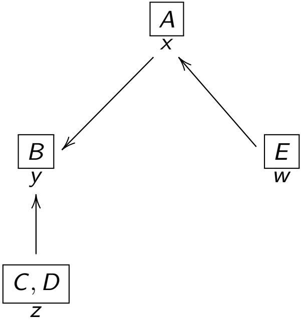

Translating from Labelled to Display

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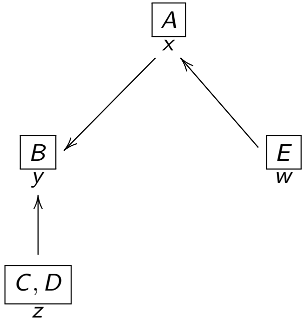
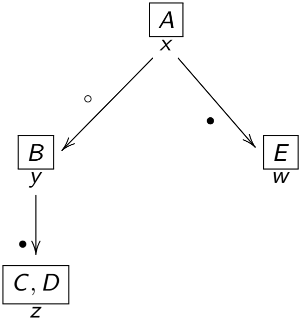
- How to translate from labelled to display? Almost as Easy!
  - (1) Pick a node
  - (2) Moving through the tree from that node label forward edges with a $\circ$ and backward edges with a $\bullet$
  - (3) Reverse all $\bullet$ edges

$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



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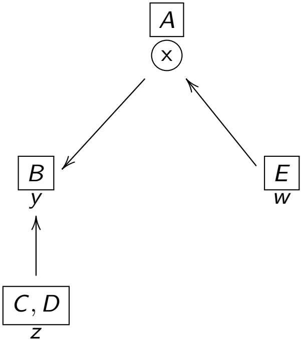

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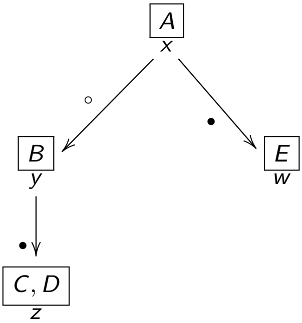

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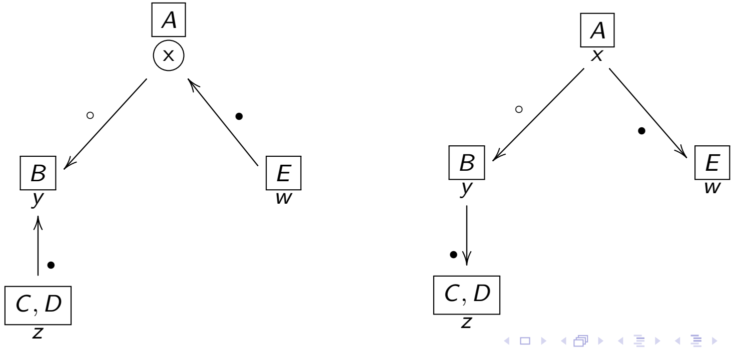

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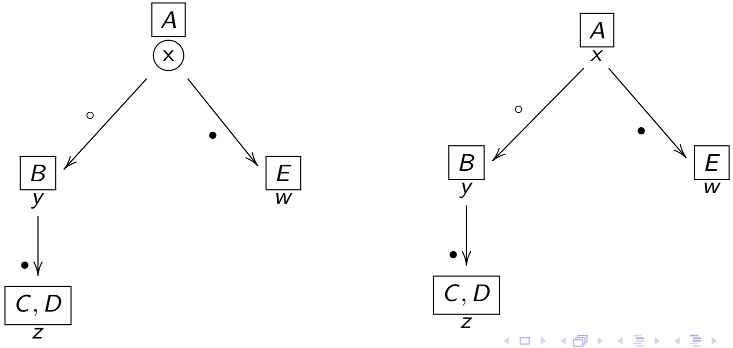

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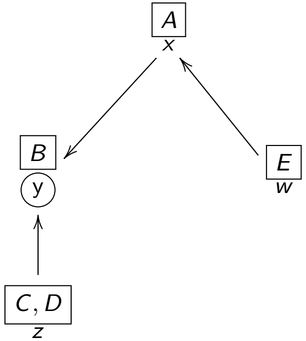

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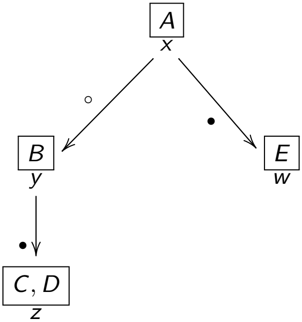

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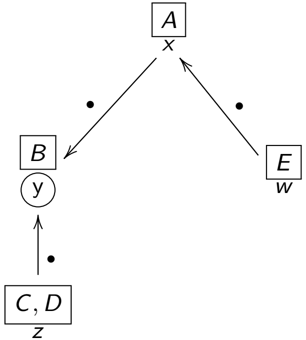

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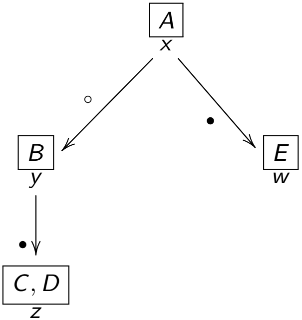

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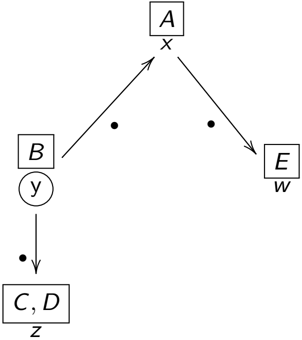

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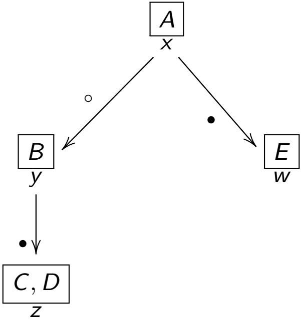
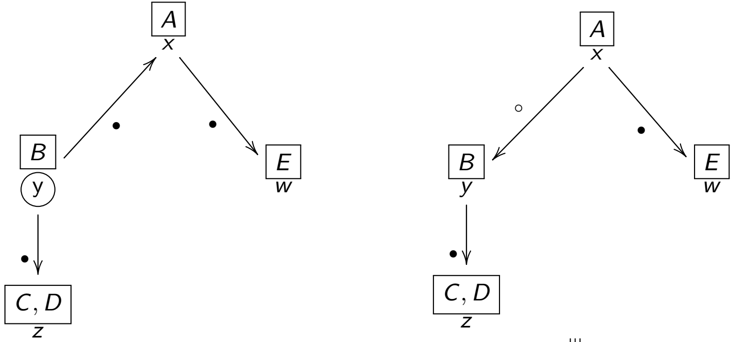


Translating from Labelled to Display... A Problem...?

- What happened? The graphs are not the same...

$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



$\equiv$

$B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}$

$\equiv$

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$

Translating from Labelled to Display... A Problem...?

- What is the relationship between

$$B \bullet \{A, \bullet\{E\}\}, \bullet\{C, D\} \text{ and } A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}?$$

Translating from Labelled to Display... A Problem...?

- What is the relationship between

$$B \bullet \{A, \bullet\{E\}\}, \bullet\{C, D\} \text{ and } A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}?$$

- An observation:

$$\frac{B \bullet \{A, \bullet\{E\}\}, \bullet\{C, D\}}{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}}\,(\mathrm{rp}) \qquad \frac{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}}{B \bullet \{A, \bullet\{E\}\}, \bullet\{C, D\}}\,(\mathrm{rf})$$

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- Better than an observation: a theorem!

Theorem

Regardless of the node chosen when translating from labelled UT sequents to display sequents, the output will be display equivalent.

Main Results

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- Theorem: Every proof of a formula in G3Kt (Labelled) is stepwise translatable to a proof in SKT (Display), and vice versa.
    - Lemma: Every sequent in an G3Kt proof **of a formula** is a labelled UT sequent.

Main Results

- Theorem: Every proof of a formula in G3Kt (Labelled) is stepwise translatable to a proof in SKT (Display), and vice versa.
  - Lemma: Every sequent in an G3Kt proof **of a formula** is a labelled UT sequent.

- Theorem: Every proof of a formula in

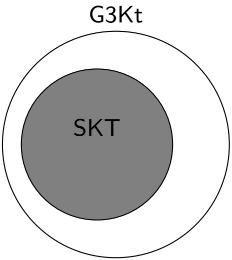
$$\mathsf{SKT} \;+\; \frac{\Gamma, \circ^{i}\{\bullet^{k}\{\Delta\}\}}{\Gamma, \bullet^{h}\{\circ^{j}\{\Delta\}\}}\,\delta SL$$

is stepwise translatable to a proof in

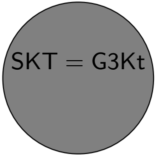
$$\mathsf{G3Kt} \;+\; \frac{\mathcal{R}, R^{i}vx, R^{k}ux, R^{h}wv, R^{j}wu, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^{h}wv, R^{j}wu, v : \Delta, u : \Delta', \Gamma}\,\lambda SL$$

Analyzing the Space of Proofs

Derivations in General:



Derivations of **formulae**:



- Every derivation **of a formula** in SKT (Display) is essentially a derivation in G3Kt (Labelled)
- Not every derivation in G3Kt (Labelled) can be transformed via our method into a derivation in SKT (Display)

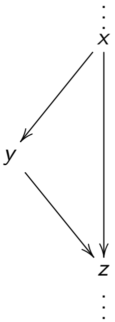
Open Question: How to Translate Labelled + Scott-Lemmon to Display?

- How can we stepwise translate in the opposite direction?

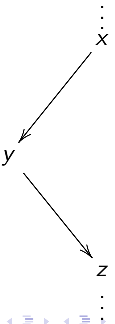
- Example:

$$\frac{\mathcal{R}, Rxz, Rxy, Ryz, \Gamma}{\mathcal{R}, Rxy, Ryz, \Gamma} \text{(Trans)}$$

Premise:



Conclusion:

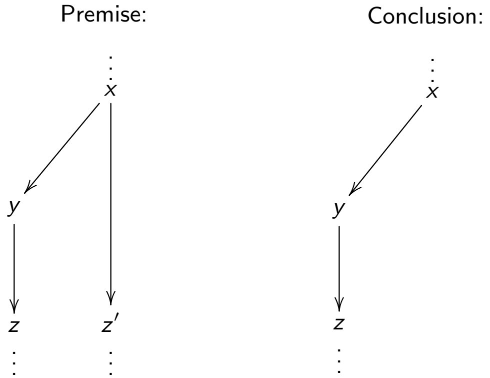


Possible Approach

- How can we stepwise translate in the opposite direction? Break the cycle?
- Example:

$$\frac{\mathcal{R}, Rxz, Rxy, Ryz, \Gamma}{\mathcal{R}, Rxy, Ryz, \Gamma}\ (\text{Trans})$$

$$\frac{\mathcal{R}', Rxz', Rxy, Ryz, \Gamma'}{\mathcal{R}, Rxy, Ryz, \Gamma}\ (\text{Trans2})$$



Future Work

- Investigate the other direction translating proofs from G3Kt $+ \lambda SL$ to proofs in SKT $+ \delta SL$
- Can cycles in labelled proofs be removed?
- Formulate formal desiderata for distinguishing internal from external calculi.
- Give stable/formal definition of internality and externality.
- Will these desiderata explain why some properties are more easily established in one type of calculus as opposed to the other?
- Can these translation methods be generalized to calculi for other logics (such as bi-intuitionistic and intermediate logics)?

Thank you for your attention