Reduction Functions

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§1. Introduction

- Proofs in modal logics are typically nonconstructive.
- A constructive method will be proposed here by reducting derivability in $L \oplus \varphi$ to derivability in L.
- Mostly, the method is not more complex than its rivals.

§2. Modal Logic: Language

Fml is the following set:

① $\{p_i : i \in \omega\} \subseteq \mathrm{Fml}$

② $\bot \in \mathrm{Fml}$.

③ If $\varphi, \chi \in \mathrm{Fml}$ then $(\neg\varphi), (\varphi \wedge \chi) \in \mathrm{Fml}$.

④ If $\varphi \in L$, also $(\Box\varphi) \in \mathrm{Fml}$.

$(\varphi \vee \chi)$ and $(\varphi \to \chi)$ are abbreviations. $\mathrm{dg}(\varphi)$ is the maximum nesting of modal operators. $\mathrm{sf}(\Delta)$ the set of subformulae of Δ, $\sharp\Delta := |\,\mathrm{sf}(\Delta)|$.

$$\Box^{<k+1}\varphi := \varphi \wedge \Box\Box^{<k}\varphi \tag{1}$$

$$\Box^{<k}\Delta := \Box^{<k} \bigwedge \Delta \tag{2}$$

§3. Normal Modal Logics

A subset $L \subseteq \mathrm{Fml}$ is a (**normal**) **modal logic** if

① L contains all tautologies of classical logic.

② L is closed under substitution.

③ L is closed under MP: If $\varphi \to \chi \in L$ and $\varphi \in L$ then $\chi \in L$.

④ $\Box(p_0 \to p_1) \to ((\Box p_0) \to (\Box p_1)) \in L$

⑤ L is closed under (MN): If $\varphi \in L$ then $\Box\varphi \in L$.

The smallest normal modal logic is called K.

§4. Some Standard Logics

$L \oplus \Delta$ denotes the smallest modal logic containing L and Δ.

$$\begin{aligned}
\mathsf{K4} &:= \mathsf{K} \oplus \Box p_0 \to \Box\Box p_0 \\
\mathsf{G} &:= \mathsf{K} \oplus \Box(\Box p_0 \to p_0) \to \Box p_0 \\
\mathsf{S4} &:= \mathsf{K} \oplus \{\Box p_0 \to \Box\Box p_0, \Box p_0 \to p_0\} \\
\mathsf{Grz} &:= \mathsf{S4} \oplus \Box(\Box(p_0 \to \Box p_0) \to p_0) \to \Box p_0 \\
\mathsf{B} &:= \mathsf{K} \oplus p_0 \to \Box\Diamond p_0 \\
\mathsf{K.alt}_1 &:= \mathsf{K} \oplus \Diamond p_0 \to \Box p_0 \\
\mathsf{T} &:= \mathsf{K} \oplus p_0 \to \Diamond p_0
\end{aligned}$$

§5. Local and Global Consequence

1. The **local consequence relation** of L. $\Delta \vdash_L \varphi$ iff φ can be proved from $\Delta \cup L$ using only (MP).
2. The **global consequence relation** of L. $\Delta \Vdash_L \varphi$ iff φ can be derived from $\Delta \cup L$ using (MP) and (MN).

§6. Reduction Functions I

Assume that $L \subseteq M$ and $\Delta \vdash_M \varphi$. Let $\Pi = \langle \pi_i : i < n \rangle$ be a Hilbert-style proof of φ from Δ. Let $X(\Delta; \varphi)$ be the set of theorems of M contained in Π. Then

$$X(\Delta; \varphi) \text{ is finite}$$
$$\Delta; X(\Delta; \varphi) \vdash_L \varphi$$

We may also assume

$$\mathrm{var}[\Pi] \subseteq \mathrm{var}[\Delta; \varphi]$$

§7. Reduction Function II

Definition 1 (Local Reduction Function) *Assume* $L \subseteq M$*.* $X : \wp(\mathrm{Fml}) \to \wp(\mathrm{Fml})$ *is a* ***local reduction function*** *from* M *to* L *if*

❶ $X(\Delta)$ *is finite if* Δ *is,*

❷ $X(\Delta) \subseteq M$*,*

❸ $\Delta \vdash_M \varphi$ *iff* $\Delta; X(\Delta; \varphi) \vdash_L \varphi$*, and*

❹ *for all* Δ*:* $\mathrm{var}[X(\Delta)] \subseteq \mathrm{var}[\Delta]$*.*

§8. Reducing Decidability

Theorem 2 *Suppose that $L \subseteq M$.*

1. *If M is decidable, there exists a* ***computable*** *reduction function to L.*
2. *If L is decidable and there exists a* ***computable*** *reduction function from M to L, M is also decidable.*

§9. Global Reduction Functions

Definition 3 (Global Reduction Function) *Assume* $L \subseteq M$*.* $X : \wp(M) \to \wp(M)$ *is a* ***global reduction function*** *from* M *to* L *if*

1. $X(\Delta)$ *is finite if* Δ *is,*
2. $X(\Delta) \subseteq M$*,*
3. $\Delta \Vdash_M \varphi$ *iff* $\Delta; X(\Delta; \varphi) \Vdash_L \varphi$*, and*
4. *for all* Δ*:* $\mathrm{var}[X(\Delta)] \subseteq \mathrm{var}[\Delta]$*.*

§10. Examples of Global Reduction Functions

The following are global reduction functions:

$$
\begin{aligned}
X_{\mathsf{4}}(\Delta) &:= \{\Box\chi \to \Box\Box\chi : \Box\chi \in \mathrm{sf}(\Delta)\} \\
X_{\mathsf{D}}(\Delta) &:= \{\neg\Box\bot\} \\
X_{\mathsf{T}}(\Delta) &:= \{\Box\chi \to \chi : \Box\chi \in \mathrm{sf}(\Delta)\} \\
X_{\mathsf{B}}(\Delta) &:= \{\neg\chi \to \Box\neg\Box\chi : \Box\chi \in \mathrm{sf}(\Delta)\} \\
X_{\mathsf{G}}(\Delta) &:= \{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box\chi) : \Box\chi \in \mathrm{sf}(\Delta)\} \\
X_{\mathsf{Grz}}(\Delta) &:= \{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box(\chi \to \Box\chi)) : \Box\chi \in \mathrm{sf}(\Delta)\} \\
X_{\mathsf{alt}_1}(\Delta) &:= \{\neg\Box\chi \to \Box\neg\chi : \Box\chi \in \mathrm{sf}(\Delta)\}
\end{aligned}
$$

The functions X_{G}, X_{Grz} are reduction functions to $\mathsf{K4}$ and $\mathsf{S4}$, respectively, all others are to K.

§11. An Example

Theorem 4 *Let L be complete for a class of finite frames that is closed under passing from the relation to its symmetric closure. Then*

$$X_{\mathsf{B}}(\Delta) := \Box^{\leq \mathrm{dg}\,\Delta}\{\neg\chi \to \Box\neg\Box\chi : \Box\chi \in \mathrm{sf}(\Delta)\} \tag{3}$$

is a local reduction function of $L \oplus \mathsf{B}$ to L.

Corollary 5 B *has the finite model property and is complete with respect to finite symmetric frames.*

Finite model property becomes a corollary of the reduction, but is implicitly used in the proof. Notice that all proofs are constructive! No need to use infinite frames.

§12. Proof

Assume that $\Delta; \Box^{\leq q} X_{\mathsf{B}}(\Delta)$ is L-consistent, where $q := \mathrm{dg}(\Delta)$. We will show that Δ is $L \oplus \mathsf{B}$-consistent. Pick a finite L-frame $\langle F, \lhd \rangle$, β and x st

$$\langle \langle F, \lhd \rangle, \beta, x \rangle \vDash \Delta; \Box^{\leq q} X_{\mathsf{B}}(\Delta) \tag{4}$$

Put $\blacktriangleleft := \lhd \cup \lhd^{\smile}$. This is an $L \oplus \mathsf{B}$-frame. By induction on χ we show that for all w reachable in $\leq q - \mathrm{dg}(\chi)$ steps from x:

$$\langle \langle F, \blacktriangleleft \rangle, \beta, w \rangle \vDash \chi \qquad \Leftrightarrow \qquad \langle \langle F, \lhd \rangle, \beta, w \rangle \vDash \chi \tag{5}$$

§13. Proof (continued)

$$(6) \qquad \langle\langle F, \blacktriangleleft\rangle, \beta, w\rangle \vDash \chi \quad \Leftrightarrow \quad \langle\langle F, \lhd\rangle, \beta, w\rangle \vDash \chi$$

The only critical step is $\chi = \Box\tau$. ($\Rightarrow$) If $x \lhd y$ then also $x \blacktriangleleft y$. ($\Leftarrow$) Assume $\langle\langle F, \blacktriangleleft\rangle, \beta, w\rangle \nvDash \Box\tau$. Then there is a v such that $w \blacktriangleleft v$ and $\langle\langle F, \blacktriangleleft\rangle, \beta, v\rangle \vDash \neg\tau$. If $w \lhd v$, we are done. Otherwise, $v \lhd w$. However, we have $\langle\langle F, \lhd\rangle, \beta, v\rangle \vDash \neg\tau \to \Box\neg\Box\neg\tau$. IH yields $\langle\langle F, \lhd\rangle, \beta, v\rangle \vDash \neg\tau$, and so $\langle\langle F, \lhd\rangle, \beta, w\rangle \vDash \neg\Box\tau$. This means $\langle\langle F, \lhd\rangle, \beta, w\rangle \nvDash \Box\tau$, as promised.

The claim now follows since Δ has depth at most q. Hence Δ is $L \oplus \mathsf{B}$-consistent. ⊠

§14. Interpolation

Definition 6 *L has* ***(local) interpolation*** *if whenever* $\varphi \vdash_L \psi$ *there exists a* χ *such that* $\mathrm{var}(\chi) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ *and* $\varphi \vdash_L \chi \vdash_L \psi$.

Say that a reduction function **splits** if

$$X(\varphi \to \chi) = X(\varphi; \chi) = X(\varphi) \cup X(\chi)$$

Theorem 7 *Suppose that there is a splitting global reduction function from* M *to* L*. Then if* L *has (local) interpolation, so does* M*.*

§15. Proof

By assumption and DT, $\Vdash_M \varphi \to \psi$. So,

$$X(\varphi \to \psi) \Vdash_L \varphi \to \psi$$

There is a k such that

$$\Box^{<k} X(\varphi \to \psi) \vdash_L \varphi \to \psi$$

Since X is splitting, $\Box^{<k} X(\varphi); \Box^{<k} X(\psi) \vdash_L \varphi \to \psi$, whence

$$\varphi; \Box^{<k} X(\varphi) \vdash_L \bigwedge \Box^{<k} X(\psi) \to \psi$$

Pick an interpolant χ:

$$\varphi; \Box^{<k} X(\varphi) \vdash_L \chi \vdash_L \bigwedge \Box^{<k} X(\psi) \to \psi$$

$\mathrm{var}(\chi) \subseteq \mathrm{var}(X(\varphi); \varphi) = \mathrm{var}(\varphi)$, $\mathrm{var}(\chi) \subseteq \mathrm{var}(X(\psi); \psi) = \mathrm{var}(\psi)$. And $\varphi \vdash_M \chi \vdash_M \psi$. $\boxtimes$

§16. Halldén-Completeness

All reduction functions shown above are splitting. Hence, interpolation follows for many standard systems. Rautenberg (1983) uses a criterion based on tableau rules, which is however quite opaque.

Definition 8 *L is said to be* ***Halldén-complete*** *if for every φ and χ disjoint in variables: If $\varphi \vdash_L \chi$ then either $\varphi \vdash_L \bot$ or $\vdash_L \chi$.*

Theorem 9 *Suppose that X is a splitting global reduction function from M to L. Then if L is Halldén-complete, so is M.*

§17. Tableaux

$\Delta_\Box := \{\chi : \Box\chi \in \Delta\}$.

$$(\neg E) \quad \frac{\Delta; \neg\neg\varphi}{\Delta; \varphi} \qquad\qquad (\wedge E) \quad \frac{\Delta; \varphi \wedge \psi}{\Delta; \varphi; \psi}$$

$$(\vee E) \quad \frac{\Delta; \neg(\varphi \wedge \psi)}{\Delta; \neg\varphi | \Delta; \neg\psi} \qquad\qquad (\Box E) \quad \frac{\Delta; \neg\Box\varphi}{\Delta_\Box; \neg\varphi}$$

Proposition 10 Δ *has a closed tableau iff* $\neg \bigwedge \Delta$ *is inconsistent in* K*.*

§18. Completeness

Rautenberg (1983) has proposed the following rule for G.

$$(G) \qquad \frac{\Delta; \neg\Box\varphi}{\Box\Delta_{\Box}; \Delta_{\Box}; \neg\varphi; \Box\varphi}$$

This rule corresponds to adding $\Box\chi \to \Box\Box\chi$ as well as $\neg\Box\varphi \to \neg\Box(\varphi \vee \neg\Box\varphi)$ above the line for certain formulae and then doing a ($\Box$E)-step. ($\neg(\varphi \vee \neg\Box\varphi) \equiv \neg\varphi \wedge \Box\varphi$.)

§19. Space Requirements of the Tableaux

Reduction functions allow to prove completeness of tableaux calculi. $||\Delta||$ is the length of Δ as a sequence, but with variables counting 1 each (so, the index i in p_i does not count). Put $\sharp\Delta := |\,\mathrm{sf}(\Delta)|$.

Proposition 11 *Let Δ be a sequence. A branch for Δ in a local tableau needs $O(\sharp\Delta \log_2 \sharp\Delta)$ space to code.*

Theorem 12 *It can be checked in $O(\sharp\Delta \log_2 \sharp\Delta)$ space whether a given set Δ of fomulae is satisfiable in K.*

§20. Improving the Space Bound

Hemaspaandra (2000) assumes only one rule:

$$\frac{\Delta; \neg\Box\varphi}{(\Delta_{\Box}; \neg\varphi)^*}$$

where Θ^* denotes a downward saturation of Θ.

At step n only the formulae of modal embedding $n+1$ become relevant. The sets of occurrences of different degree are disjoint. Hence we get

Theorem 13 (Hemaspaandra (2000)) *It can be checked using $O(\sharp\Delta)$ space whether a given sequence Δ of formulae is satisfiable in* K*.*

§21. Global Space Bounds via Reduction Functions

Lemma 14 *Let $L \supseteq \mathsf{K}$. Let X be a global reduction function from L to K. If X is polynomial in $\sharp\Delta$, L is globally EXPTIME.*

Theorem 15 *Let $L \supseteq \mathsf{K}$. Suppose that X is a global reduction function from L to K. Suppose further that there is a finite set Θ of L-theorems such that $X(\Delta)$ consists of some or all substitution instances of Θ by members of* $\mathrm{sf}(\Delta)$ *for its variables. Then L is globally in EXPTIME with respect to any of the measures.*

Let n be the number of variables in Θ. Then we say X is n-**analytic** with **skeletal set** Θ.

§22. Local Space Bounds I

Definition 16 *Suppose that X is an n-analytic global reduction from M to L with skeletal set Θ and that ρ is a function from sets of formulae to natural numbers. If*

$$Y(\Delta) := \{\Box^{<\rho(\Delta)+1} \bigwedge X(\Delta)\} \tag{7}$$

*is a local reduction function from M to L, Y is called n-**analytic** with **skeletal set** Θ and **depth reduction function** ρ.*

§23. Local Space Bounds II

Theorem 17 *Let $n > 0$. Suppose that Y is an n-analytic local reduction function from L to* K *with skeletal set Θ and depth reduction function ρ with $\rho(\Delta) \leq \sharp\Delta$. Then if Δ is L-unsatisfiable then a closing* K*-tableau for $\Delta; Y(\Delta)$ can be computed using $O((\sharp\Delta)^n \operatorname{dg}(\Delta)))$ space.*

Theorem 18 *Let M be a union of any the following logics:* K, KT, KB, $\mathsf{K.alt}_1$, KD. *Then M is locally in $O(\sharp\Delta \log \sharp\Delta)$-space.*

§24. Global to Local Reduction

One can also reduce the global consequence to the local. Consider $h : \wp(\mathrm{Fml}) \to \mathbb{N}$ such that

$$\Delta \Vdash_L \varphi \quad \Leftrightarrow \quad \Box^{<h(\Delta;\varphi)}\Delta \vdash_L \varphi$$

The depth reduction function for K is exponential.

Theorem 19 *Let $p := \sharp\Delta$ and $q := \sharp\varphi$. Then*

$$\Delta \Vdash_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad \Box^{<2^p+q}\Delta \vdash_{\mathsf{K}} \varphi$$

The bound cannot be significantly reduced. Eg if $L \subseteq \mathsf{K.alt}_1$, then there are infinitely many Δ such that

$$h_L(\Delta) \geq 2^{\sqrt{\sharp\Delta}}$$

§25. Transitive Logics

For L transitive, $\Delta \Vdash_L \varphi$ iff $\Delta; \Box\Delta \vdash_L \varphi$. This transformation of problems is linear; thus, the local/global distinction collapses. Hemaspaandra's method can be used to obtain an easy proof for the following result.

Theorem 20 (Nguyen (1999)) *Local satisfiability in* $\mathsf{K4}$ *can be checked in* $O(\sharp\Delta \log \sharp\Delta)$ *space.*

Theorem 21 *Let* L *be any union of* KT, KB, KD, $\mathsf{K.alt_1}$, $\mathsf{K4}$, KD, $\mathsf{K4.G}$ *and* $\mathsf{K4.Grz}$. *Then* L *is locally in* $O(\sharp\Delta \log \sharp\Delta)$*-space.*

§26. Further Results

- The method is straightforwardly generalized to polymodal logics.
- Extensions of S4.3 are cofinal subframe logics (see Chagrov & Zakharyaschev (1997)). For $L \supseteq$ S4.3 only a single tableau needs to be computed. Hence, L is in NP and NP-complete if consistent (see Spaan (1993)).
- Satisfiability in tense logic is in $O(\sharp\Delta \log \sharp\Delta)$.
- PDL with converse is EXPTIME-complete (see de Giacomo (1996)).
- Many splitting axioms preserve complexity bounds above K4 and S4 (see Kracht (1993)). Eg (above K4): .1, .2 and (above S4): .Dum.

- Using the standard Gödel-translation, which is linear, the space bound $O(\sharp\Delta \log \sharp\Delta)$ for intuitionistic propositional logic given in Hudelmaier (1993) can also be established.

§27. Conclusion

① Reduction functions provide constructive proofs of the finite model property, decidability, interpolation and Halldén-completeness of many standard systems.

② The method is uniform and requires only to establish the reduction. Everything is else (eg interpolation, Halldén-completeness) is a straightforward application of general theorems.

③ The best known complexity bounds can be established for the standard systems.