Internalizing labels in BI logics

Meeting TICAMORE Marseille

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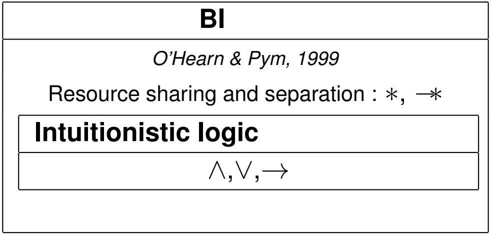
Introduction

BI logics



BI

O'Hearn & Pym, 1999

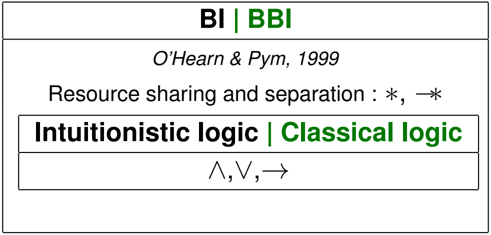
Resource sharing and separation : $*$, $-\!*$

Intuitionistic logic

$\wedge, \vee, \rightarrow$

Introduction

BI logics



BI | BBI

O'Hearn & Pym, 1999

Resource sharing and separation : $\ast$, $\mathrel{-\!\!\ast}$

Intuitionistic logic | Classical logic

$\wedge, \vee, \rightarrow$

Introduction

BBI semantics

- R set of resources, $\bullet$ composition, e neutral element
  - ↪ $r \vDash \phi$
  - ↪ resources : knowledge, space, general context...

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BBI semantics

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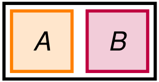
  - $\hookrightarrow$ $r \models \phi$
  - $\hookrightarrow$ resources : knowledge, space, general context...
- $r \models A * B$ iff $\exists r_1, r_2 \in R$ such that $r = r_1 \bullet r_2$ and $r_1 \models A$ and $r_2 \models B$

$A * B$



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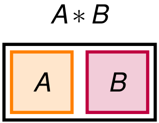
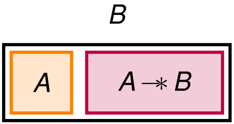


$A * B$

A B

- $r \vDash A \mathrel{-\!\!*} B$ iff $\forall r' \in R$, if $r' \vDash A$ then $r \bullet r' \vDash B$



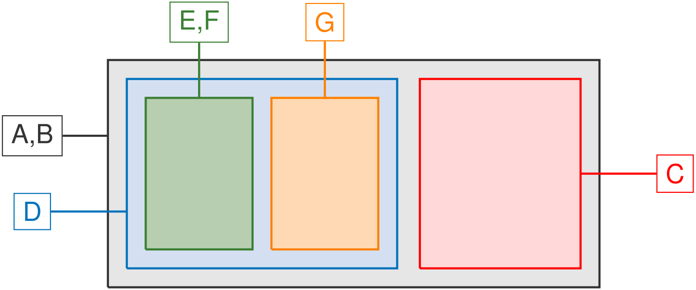
B

A $A \mathrel{-\!\!*} B$

Introduction

BBI example

$$((((E \wedge F) * G) \wedge D) * C) \wedge A \wedge B$$



Introduction

Hybrid logic

Modal logic (especially temporal)

- $\Box\phi$: *For all* states that follow, ϕ is valid
- $\Diamond\phi$: *There exists* a state that follow where ϕ is valid

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$\hookrightarrow$ Quantifiers over states, no way to capture a precise state.

Prior, 1967 / Blackburn, 2006

$\Rightarrow$ Hybrid logic : addition of state labels in the syntax

- $@_s(\phi)$: ϕ is valid at state *s*

Introduction

Motivations

Why not do the same with BBI ?

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$\hookrightarrow$ Hybrid Resource Logic : BBI + location operators from Hybrid Logic

$\Rightarrow$ Extends **expressiveness** (similarly to Hybrid Logics)

$\Rightarrow$ Allows **axiomatisation** of BBI properties

Introduction

Contributions

- A new logic to reason on sharing and separating resources
- Syntax including location operators with resource labels
- Weaker semantics than BBI, added properties through axioms
- Axioms allow to recapture BBI expressiveness **and** some variants
- Extended expressiveness through location operator
- Tableau method without labels (soundness/completeness)

HRL logic

HRL logic

Syntax

Set of **propositional symbols** : *Prop*

Set of resource symbols or **nominals** : *Nom*

HRL language is defined by the following grammar :

$$
\begin{array}{llll}
X ::= p \in Prop & | \top & | \bot & \\
\quad | \neg X & | X \wedge X & | X \vee X & | X \rightarrow X \\
\quad | \mathrm{I} & | X * X & | X \mathrel{-\!\!*} X & | X \mathrel{*\!\!-} X \\
\quad | i \in Nom & | @_i(X) & &
\end{array}
$$

Note : differentiation between $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$ is necessary because composition won't always be commutative.

HRL logic

Semantics

Definition (Weak resource structure)

*A **weak resource structure** associated to Nom is a triple $\mathcal{R} = (\bullet, e, \sim)$ such that:*

- *$e \in$ Nom ;*
- *$\bullet$: Nom $\times$ Nom $\rightharpoonup$ Nom ;*
- *$\sim$ is an equivalence relation on Nom compatible with $\bullet$.*

Definition (Interpretation)

*An **interpretation** of Prop for $\mathcal{R}$ is a function $[\![\cdot]\!]$: Prop $\to \mathbb{P}(Nom)$ which is monotone on Prop, which means for all $p \in$ Prop, for all $r, r' \in$ Nom, if $r \sim r'$ and $r \in [\![p]\!]$ then $r' \in [\![p]\!]$.*

HRL logic

Semantics

Definition (Model)

A ***model*** *of HRL is a triple* $\mathcal{K} = (\mathcal{R}, [\![\cdot]\!], \models_{\mathcal{K}})$ *where* $\mathcal{R} = (\bullet, e, \sim)$ *is a weak resource structure on Nom,* $[\![\cdot]\!]$ *is an interpretation of Prop for* $\mathcal{R}$ *and* $\models_{\mathcal{K}} \subseteq \mathcal{L} \times$ *Nom is defined by :*

- $r \models_{\mathcal{K}} p$ *iff* $r \in [\![p]\!]$
- $r \models_{\mathcal{K}} \phi \wedge \psi$ *iff* $r \models_{\mathcal{K}} \phi$ *and* $r \models_{\mathcal{K}} \psi$
- $r \models_{\mathcal{K}} \phi * \psi$ *iff there exist* $r', r'' \in$ *Nom such that* $r' \bullet r'' \downarrow$ *and* $r' \bullet r'' \sim r$ *and* $r' \models_{\mathcal{K}} \phi$ *and* $r'' \models_{\mathcal{K}} \psi$
- $r \models_{\mathcal{K}} \phi \mathrel{-\!\!*} \psi$ *iff for all* $r' \in$ *Nom such that* $r \bullet r' \downarrow$ *and* $r' \models_{\mathcal{K}} \phi$*, we have* $r \bullet r' \models_{\mathcal{K}} \psi$
- $r \models_{\mathcal{K}} \phi \mathrel{*\!\!-} \psi$ *iff for all* $r' \in$ *Nom such that* $r' \bullet r \downarrow$ *and* $r' \models_{\mathcal{K}} \phi$*, we have* $r' \bullet r \models_{\mathcal{K}} \psi$
- $r \models_{\mathcal{K}} i$ *iff* $r \sim i$
- $r \models_{\mathcal{K}} @_i(\phi)$ *iff* $i \models_{\mathcal{K}} \phi$

HRL logic

HBBI logic

Definition (HBBI logic)

HBBI logic *is the fragment of HRL where the following axioms are valid for any* $i, j, k \in Nom$ *:*

$$
\begin{aligned}
(BI)_n &\equiv @_i(i * \mathrm{I}) \\
(BI)_c &\equiv j * k \rightarrow k * j \\
(BI)_a &\equiv j * (k * l) \rightarrow (j * k) * l
\end{aligned}
$$

Theorem (Semantic equivalence between HBBI and BBI)

Let ϕ *be a BI formula.*
If any model of BBI is built on Nom, then $\vDash_{BBI} \phi$ *iff* $\vDash_{HBBI} \phi$*.*

Note : in HBBI, $A \mathrel{-\!\!*} B \equiv A \mathrel{*\!\!-} B$

A tableau method for HRL

A tableau method for HRL

Formulae and SS

Definition (Labelled formulae, Set of statements)

*A **labelled formula** is a pair* $(\mathbb{S}, \Phi)$ *with* $\mathbb{S} \in \{\mathbb{T}, \mathbb{F}\}$ *and* Φ *a HRL-formula of the form* $\Phi = @_x(\phi)$ *where* $x \in$ *Nom et* $\phi \in \mathcal{L}$*.*

We note $\mathbb{S}\ @_x(\phi)$ *a labelled formula* $(\mathbb{S}, @_x(\phi))$*.*

*A **Set of Statements** or SS, noted* $\mathcal{F}$ *is a set of labelled formulae. The **alphabet** of* $\mathcal{F}$*, noted* $\mathcal{A}(\mathcal{F})$ *is the set of nominals appearing in* $\mathcal{F}$*.*

A tableau method for HRL

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$$\mathbb{S}x : \phi \quad \rightsquigarrow \quad \mathbb{S}\ @_x(\phi)$$

BI labelled tableaux HRL unlabelled tableaux

A tableau method for HRL

Additive Rules

$$\frac{\mathbb{T}\ @_x(\phi \wedge \psi)}{\mathbb{T}\ @_x(\phi),\ \mathbb{T}\ @_x(\psi)}\ \langle \mathbb{T}\wedge \rangle \qquad \frac{\mathbb{F}\ @_x(\phi \wedge \psi)}{\mathbb{F}\ @_x(\phi) \mid \mathbb{F}\ @_x(\psi)}\ \langle \mathbb{F}\wedge \rangle$$

$$\frac{\mathbb{T}\ @_x(\phi \vee \psi)}{\mathbb{T}\ @_x(\phi) \mid \mathbb{T}\ @_x(\psi)}\ \langle \mathbb{T}\vee \rangle \qquad \frac{\mathbb{F}\ @_x(\phi \vee \psi)}{\mathbb{F}\ @_x(\phi),\ \mathbb{F}\ @_x(\psi)}\ \langle \mathbb{F}\vee \rangle$$

$$\frac{\mathbb{T}\ @_x(\phi \to \psi)}{\mathbb{F}\ @_x(\phi) \mid \mathbb{T}\ @_x(\psi)}\ \langle \mathbb{T}\to \rangle \qquad \frac{\mathbb{F}\ @_x(\phi \to \psi)}{\mathbb{T}\ @_x(\phi),\ \mathbb{F}\ @_x(\psi)}\ \langle \mathbb{F}\to \rangle$$

$$\frac{\mathbb{T}\ @_x(\neg\phi)}{\mathbb{F}\ @_x(\phi)}\ \langle \mathbb{T}\neg \rangle \qquad \frac{\mathbb{F}\ @_x(\neg\phi)}{\mathbb{T}\ @_x(\phi)}\ \langle \mathbb{F}\neg \rangle$$

x is a nominal.

A tableau method for HRL

Multiplicative Rules

$$\dfrac{\mathbb{T}\ @_x(\phi * \psi)}{\mathbb{T}\ @_{c_i}(\phi),\ \mathbb{T}\ @_{c_j}(\psi),\ \mathbb{T}\ @_x(c_i * c_j)}\ \langle \mathbb{T}* \rangle \qquad \dfrac{\mathbb{F}\ @_x(\phi * \psi), \mathbb{T}\ @_x(y * z)}{\mathbb{F}\ @_y(\phi) \mid \mathbb{F}\ @_z(\psi)}\ \langle \mathbb{F}* \rangle$$

$$\dfrac{\mathbb{T}\ @_x(\phi \mathrel{-\!\!*} \psi), \mathbb{T}\ @_z(x * y)}{\mathbb{F}\ @_y(\phi) \mid \mathbb{T}\ @_z(\psi)}\ \langle \mathbb{T}\mathrel{-\!\!*} \rangle \qquad \dfrac{\mathbb{F}\ @_x(\phi \mathrel{-\!\!*} \psi)}{\mathbb{T}\ @_{c_i}(\phi),\ \mathbb{F}\ @_{c_j}(\psi),\ \mathbb{T}\ @_{c_j}(x * c_i)}\ \langle \mathbb{F}\mathrel{-\!\!*} \rangle$$

$$\dfrac{\mathbb{T}\ @_x(\phi \mathrel{*\!\!-} \psi), \mathbb{T}\ @_z(y * x)}{\mathbb{F}\ @_y(\phi) \mid \mathbb{T}\ @_z(\psi)}\ \langle \mathbb{T}\mathrel{*\!\!-} \rangle \qquad \dfrac{\mathbb{F}\ @_x(\phi \mathrel{*\!\!-} \psi)}{\mathbb{T}\ @_{c_i}(\phi),\ \mathbb{F}\ @_{c_j}(\psi),\ \mathbb{T}\ @_{c_j}(c_i * x)}\ \langle \mathbb{F}\mathrel{*\!\!-} \rangle$$

x, y, z are nominals and c_i, c_j are new nominals.

A tableau method for HRL

Label Rules

$$\frac{\mathbb{S}\ @_x(@_y(\phi))}{\mathbb{S}\ @_y(\phi)}\ \langle @ \rangle \qquad \frac{}{\mathbb{T}\ @_x(x)}\ \langle i_r \rangle$$

$$\frac{\mathbb{T}\ @_x(y)}{\mathbb{T}\ @_y(x)}\ \langle i_s \rangle \qquad \frac{\mathbb{S}\ @_x(\phi), \mathbb{T}\ @_x(y)}{\mathbb{S}\ @_y(\phi)}\ \langle i_t \rangle$$

$$\frac{\mathbb{S}\ @_x(\phi[y * z])}{\mathbb{S}\ @_x(\phi[y * z / c_i]),\ \mathbb{T}\ @_{c_i}(y * z)}\ \langle i+ \rangle \qquad \frac{\mathbb{S}\ @_x(\phi[y]), \mathbb{T}\ @_y(z * t)}{\mathbb{S}\ @_x(\phi[y / z * t])}\ \langle i- \rangle$$

$$\frac{\mathbb{S}\ @_x(\phi[y]), \mathbb{T}\ @_y(z)}{\mathbb{S}\ @_x(\phi[y / z])}\ \langle i_p \rangle$$

x, y, z, t are nominals and c_i is a new nominal.

A tableau method for HRL

Closure

A tableau for a formula ϕ is a tableau for $\{\mathbb{F}\ @_{c_1}(\phi)\}$ where c_1 is a nominal not appearing in ϕ.

Definition (Closure)

A SS $\mathcal{F}$ is ***closed*** *if one of the following is verified (for $\phi \in \mathcal{L}$ and $x \in Nom$) :*

1. $\mathbb{T}\ @_x(\phi) \in \mathcal{F}$ *and* $\mathbb{F}\ @_x(\phi) \in \mathcal{F}$
2. $\mathbb{T}\ @_x(\bot) \in \mathcal{F}$
3. $\mathbb{F}\ @_x(\top) \in \mathcal{F}$

A SS is ***opened*** *if it's not closed*

A tableau is ***closed*** *if all its branches (its SS) are closed.*

A ***tableau-proof*** *for a formula ϕ is a closed tableau for ϕ.*

A tableau method for HRL

Properties of the method

Theorem (Soundness)

If there exists a proof for a HRL-formula ϕ, then it is valid.

Proof.

Through realisability of branches. □

Theorem (Completeness)

Let ϕ be a HRL-formula. If ϕ is valid, then there is a proof of ϕ.

Proof.

Through construction of a Hintikka branch and extraction of counter-model from this saturated branch. □

A tableau method for HRL

Tableau example

$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$$

A tableau method for HRL

Tableau example

$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$

$$\boxed{\dfrac{\mathbb{F}\ @_x(\phi \to \psi)}{\mathbb{T}\ @_x(\phi),\ \mathbb{F}\ @_x(\psi)}\ \langle \mathbb{F} \to \rangle}$$

A tableau method for HRL

Tableau example

$$\begin{array}{c} \mathbb{F}\ @_{c_1}(@_i(A)\wedge(i*B)\to A*B) \\ | \\ \mathbb{T}\ @_{c_1}(@_i(A)\wedge(i*B)) \\ \mathbb{F}\ @_{c_1}(A*B) \end{array}$$

$$\boxed{\frac{\mathbb{F}\ @_x(\phi\to\psi)}{\mathbb{T}\ @_x(\phi),\ \mathbb{F}\ @_x(\psi)}\ \langle\mathbb{F}\to\rangle}$$

A tableau method for HRL

Tableau example

$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$$

$$|$$

$$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$$

$$\mathbb{F}\ @_{c_1}(A * B)$$

$$\boxed{\frac{\mathbb{T}\ @_x(\phi \wedge \psi)}{\mathbb{T}\ @_x(\phi),\ \mathbb{T}\ @_x(\psi)}\ \langle \mathbb{T}\wedge \rangle}$$

A tableau method for HRL

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$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \rightarrow A * B)$$

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$$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$$
$$\mathbb{F}\ @_{c_1}(A * B)$$

$$|$$

$$\mathbb{T}\ @_{c_1}(@_i(A))$$
$$\mathbb{T}\ @_{c_1}(i * B)$$

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Tableau example

$$\begin{array}{c}
\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B) \\
| \\
\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B)) \\
\mathbb{F}\ @_{c_1}(A * B) \\
| \\
\mathbb{T}\ @_{c_1}(@_i(A)) \\
\mathbb{T}\ @_{c_1}(i * B)
\end{array}$$

$$\boxed{\frac{\mathbb{S}\ @_x(@_y(\phi))}{\mathbb{S}\ @_y(\phi)}\ \langle @ \rangle}$$

A tableau method for HRL

Tableau example

$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \rightarrow A * B)$$

$$|$$

$$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$$
$$\mathbb{F}\ @_{c_1}(A * B)$$

$$|$$

$$\mathbb{T}\ @_{c_1}(@_i(A))$$
$$\mathbb{T}\ @_{c_1}(i * B)$$

$$|$$

$$\mathbb{T}\ @_i(A)$$

$$\boxed{\frac{\mathbb{S}\ @_x(@_y(\phi))}{\mathbb{S}\ @_y(\phi)}\ \langle @ \rangle}$$

A tableau method for HRL

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\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \rightarrow A * B) \\
\vert \\
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\mathbb{F}\ @_{c_1}(A * B) \\
\vert \\
\mathbb{T}\ @_{c_1}(@_i(A)) \\
\mathbb{T}\ @_{c_1}(i * B) \\
\vert \\
\mathbb{T}\ @_i(A)
\end{array}$$

$$\boxed{\frac{\mathbb{T}\ @_x(\phi * \psi)}{\mathbb{T}\ @_{c_i}(\phi),\ \mathbb{T}\ @_{c_j}(\psi),\ \mathbb{T}\ @_x(c_i * c_j)}\ \langle\mathbb{T}*\rangle}$$

A tableau method for HRL

Tableau example

$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$$

$$|$$

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$$\mathbb{T}\ @_{c_1}(@_i(A))$$
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$$|$$

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$$|$$

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$$\boxed{\dfrac{\mathbb{F}\ @_x(\phi * \psi), \mathbb{T}\ @_x(y * z)}{\mathbb{F}\ @_y(\phi) \mid \mathbb{F}\ @_z(\psi)}\ \langle \mathbb{F} * \rangle}$$

A tableau method for HRL

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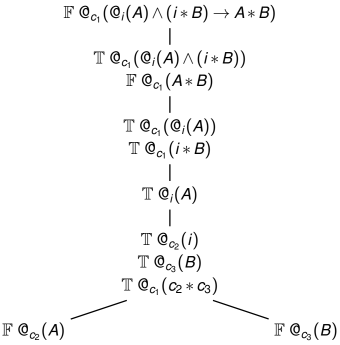

$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$

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A tableau method for HRL

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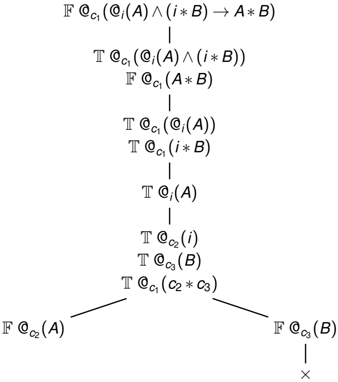


$$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$$

$$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$$

$$\mathbb{F}\ @_{c_1}(A * B)$$

$$\mathbb{T}\ @_{c_1}(@_i(A))$$

$$\mathbb{T}\ @_{c_1}(i * B)$$

$$\mathbb{T}\ @_i(A)$$

$$\mathbb{T}\ @_{c_2}(i)$$

$$\mathbb{T}\ @_{c_3}(B)$$

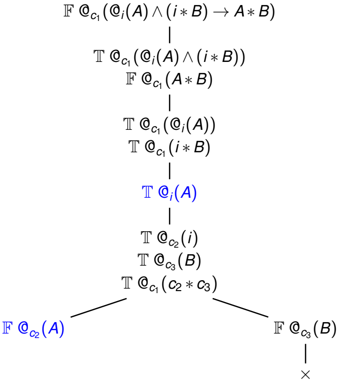
$$\mathbb{T}\ @_{c_1}(c_2 * c_3)$$

$$\mathbb{F}\ @_{c_2}(A) \qquad \mathbb{F}\ @_{c_3}(B)$$

$$\times$$

A tableau method for HRL

Tableau example



$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \rightarrow A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$

$\mathbb{F}\ @_{c_1}(A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A))$

$\mathbb{T}\ @_{c_1}(i * B)$

$\mathbb{T}\ @_i(A)$

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$\mathbb{T}\ @_{c_3}(B)$

$\mathbb{T}\ @_{c_1}(c_2 * c_3)$

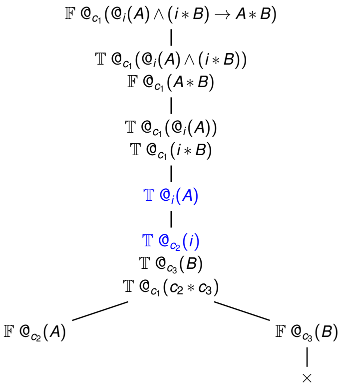
$\mathbb{F}\ @_{c_2}(A)$

$\mathbb{F}\ @_{c_3}(B)$

$\times$

A tableau method for HRL

Tableau example



$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$

$\mathbb{F}\ @_{c_1}(A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A))$

$\mathbb{T}\ @_{c_1}(i * B)$

$\mathbb{T}\ @_i(A)$

$\mathbb{T}\ @_{c_2}(i)$

$\mathbb{T}\ @_{c_3}(B)$

$\mathbb{T}\ @_{c_1}(c_2 * c_3)$

$\mathbb{F}\ @_{c_2}(A)$

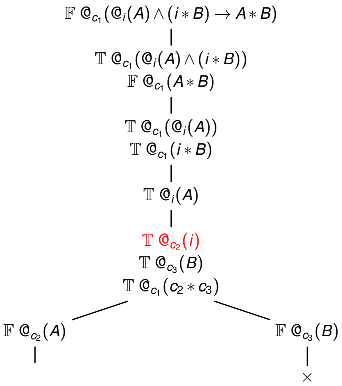
$\mathbb{F}\ @_{c_3}(B)$

$\times$

$$\boxed{\frac{\mathbb{S}\ @_x(\phi), \mathbb{T}\ @_x(y)}{\mathbb{S}\ @_y(\phi)}\ \langle i_t \rangle}$$

A tableau method for HRL

Tableau example



$\mathbb{F}\ @_{c_1}(@_i(A) \wedge (i * B) \to A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A) \wedge (i * B))$

$\mathbb{F}\ @_{c_1}(A * B)$

$\mathbb{T}\ @_{c_1}(@_i(A))$

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$\mathbb{T}\ @_{c_1}(c_2 * c_3)$

$\mathbb{F}\ @_{c_2}(A)$

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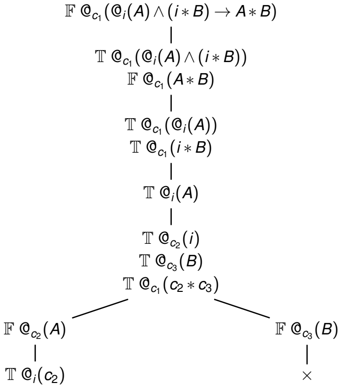

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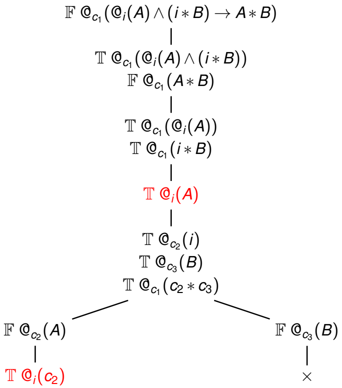

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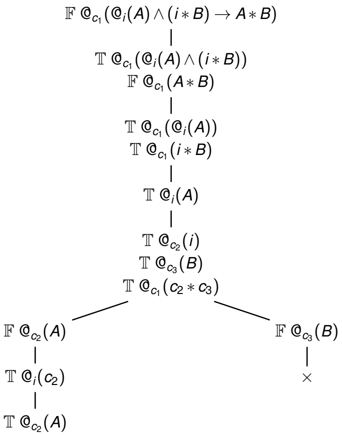

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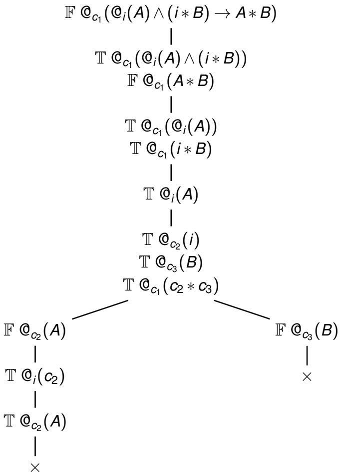

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A tableau method for HRL

HBBI tableaux

HBBI axioms

$$(BI)_n \equiv @_i(i * \mathrm{I})$$

$$(BI)_c \equiv j * k \rightarrow k * j$$

$$(BI)_a \equiv j * (k * l) \rightarrow (j * k) * l$$

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$$\rightsquigarrow$$

HBBI tableaux rules

$$\frac{}{\mathbb{T}\ @_x(x * \mathrm{I})}\ \langle BI_n\rangle$$

$$\frac{\mathbb{T}\ @_x(y * z)}{\mathbb{T}\ @_x(z * y)}\ \langle BI_c\rangle$$

$$\frac{\mathbb{T}\ @_x(y * (z * t))}{\mathbb{T}\ @_x((y * z) * t)}\ \langle BI_a\rangle$$

- Soundness is conserved
- Completeness have to be studied

Expressiveness

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Why use HRL ?

Expressiveness

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⇒ Extended expressiveness

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Why use HRL ?

⇒ Extended expressiveness

⇒ Modular logic : addition of axioms

Expressiveness

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- Equivalence relation : two resources are equivalent if they represent the same sum of money.
  E.G. : $e_2 \sim e_1 \bullet c_{50} \bullet c_{20} \bullet c_{20} \bullet c_{10}$

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4. $e_1 \models Obj_{(0.30)} * \top$

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More expressiveness with HBBI

We develop the same example with HBBI (so that everything we've stated is still valid). The set of nominal is the set of elementary resources ($Nom = Res$).

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Even further :

$$e_2 \bullet e_1 \vDash (Obj_{(0.30)} * x) \wedge @_x(Obj_{(1.70)} * y)$$

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BBI extensions and restrictions

$$\text{HRL} + (BI)_n + (BI)_c + (BI)_a \quad = \quad \text{HBBI}$$

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Expressiveness

BBI extensions and restrictions

HRL + $(BI)_n$ + $(BI)_c$ + $(BI)_a$ = HBBI

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HBBI + (Inv) = HBBI with invertible resources

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Note : in $(i * x) \wedge \mathrm{I}$, x is the invert of i.

Conclusion and perspectives

Contribution :

- A new logic, HRL, weaker than BBI but with internalized labels
- A tableau method for HRL (sound and complete)
- An extension of HRL, HBBI, that matches exactly BBI

Perspectives :

- Extended expressiveness of BI logics
- Easy extensions and restrictions to new logics, with tableau method