Equivalence between internal and labelled sequent calculi for Lewis' conditional logic $\mathbb{V}$

Marianna Girlando, Sara Negri, Nicola Olivetti

Aix Marseille Université, Laboratoire des Sciences de l'Information et des Systèmes;
University of Helsinki, Department of Philosophy;
Aix Marseille Université, Laboratoire des Sciences de l'Information et des Systèmes

TICAMORE Meeting
15 - 18 November, Marseille

Equivalence results

Why?

- Investigate the interrelations between different proof systems;
- Deeper understanding of the systems;
- Transfer proof-theoretic and model theoretic results between the calculi.

Partial references

- Fitting (2011);
- Poggiolesi (2011);
- Goré and Ramanayake (2012).

Our case study

Conditional logic
$\mathbb{V}$

G3V
Labelled sequent calculs

$\mathcal{I}^{i}_{\mathbb{V}}$
Internal sequent calculus

References

- Negri and Olivetti (2015);
- Olivetti and Pozzato (2015); Girlando, Lellmann, Olivetti and Pozzato (2016).

Outline

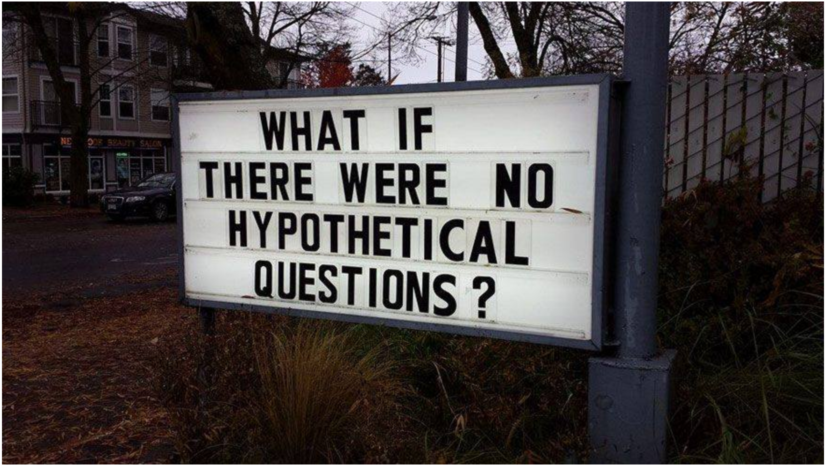
(1) Backgrounds and proof systems for $\mathbb{V}$

(2) Translation: from $\mathcal{I}^i_{\mathbb{V}}$ to **G3V**

(3) Inverse translation: from **G3V** to $\mathcal{I}^i_{\mathbb{V}}$

(4) Conclusions

Backgrounds



Conditional operators

- Counterfactual conditional operator: $A > B$

  "If A were the case, then B would have been the case"

- Comparative plausibility operator: $A \preccurlyeq B$

  "A is at least as plausible as B"

- The two operators are interdefinable:

$$A > B \;\equiv\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$$

$$A \preccurlyeq B \;\equiv\; ((A \vee B) > \bot) \vee \neg((A \vee B) > \neg A)$$

Language

$$A, B ::= P \mid \bot \mid A \rightarrow B \mid A \preccurlyeq B$$

Axioms and inference rules

Axioms and inference rules of classical propositional logic;

(CPR) if $\vdash A \preccurlyeq B$ then $\vdash B \rightarrow A$

(CPA) $(A \preccurlyeq (A \vee B)) \vee (B \preccurlyeq (A \vee B))$

(TR) $(A \preccurlyeq B) \wedge (B \preccurlyeq C) \rightarrow (A \preccurlyeq C)$

(CO) $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

Neighbourhood models for $\mathbb{V}$

A *neighbourhood model* $\mathcal{M} = \langle W, I, [\![\ ]\!] \rangle$ consists of:

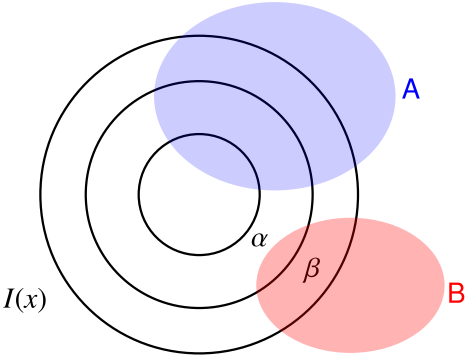
- W, non-empty set of elements;
- $I : W \rightarrow \mathcal{P}(\mathcal{P}(W))$, function assigning to each x a set $I(x)$. Let $\alpha, \beta, ...$ be elements of $I(x)$;
- $[\![\ ]\!]$: *Atm* $\rightarrow \mathcal{P}(W)$, propositional evaluation.

A model for $\mathbb{V}$ satisfies:

- *(Non-emptiness)* for each $\alpha \in I(x)$, $\alpha \neq \emptyset$;
- *(Nesting)* for each $\alpha, \beta \in I(x)$, $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$.

Comparative plausibility

- $\alpha \Vdash^{\exists} A$ *iff* $\exists y \in \alpha\, (y \Vdash A)$
- $x \Vdash A \preccurlyeq B$ *iff* $\forall \alpha \in I(x)(\alpha \Vdash^{\exists} B$ *implies* $\alpha \Vdash^{\exists} A)$



Proof systems

Labelled sequent calculus Negri and Olivetti (2015)

Two kinds of labels

- labels for worlds: $x, y, z \dots$
- labels for neighbourhoods: $a, b, c \dots$

Expressions employed in the calculus

- $a \in I(x)$
- $x \in a$
- $a \subseteq b$
- $x : A$
- $a \Vdash^{\exists} A \;\equiv\; \exists x\,(x \in a \text{ and } x \Vdash A)$
- $x : A \preccurlyeq B \;\equiv\; \forall a \in I(x)(a \Vdash^{\exists} B \text{ implies } a \Vdash^{\exists} A)$

Rules of **G3V** (1)

Initial sequents

$$x : p, \Gamma \Rightarrow \Delta, x : p \qquad\qquad x : \bot, \Gamma \Rightarrow \Delta$$

Rules for local forcing

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ L{\Vdash^{\exists}}\ (x \text{ fresh}) \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ R{\Vdash^{\exists}}$$

Propositional rules

$$\frac{\Gamma \Rightarrow \Delta, x : A}{x : \neg A, \Gamma \Rightarrow \Delta}\ L\neg \qquad \frac{x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg A}\ R\neg$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ R\to$$

Rules of **G3V** (2)

Rules for comparative plausibility

$$\frac{a \Vdash^{\exists} B, a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B}\ R\preccurlyeq\ (a \text{ new})$$

$$\frac{a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} B \quad a \Vdash^{\exists} A, a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}{a \in I(x), x : A \preccurlyeq B, \Gamma \Rightarrow \Delta}\ L\preccurlyeq$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ Tr \qquad \frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Rule for nesting

$$\frac{a \subseteq b, a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta \qquad b \subseteq a, a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ Nes$$

Example

Derivation of $A \preccurlyeq B \vee B \preccurlyeq A$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{(AX)}{a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B, y : B}
          }{a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B} \; R\Vdash^{\exists}
        }{a \subseteq b, a \in I(x), b \in I(x), y \in a, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B} \; L\subseteq
      }{a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B} \; L\Vdash^{\exists}
      \qquad (*)
    }{a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B} \; Nes
  }{a \in I(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A} \; R\preccurlyeq
}{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A} \; R\preccurlyeq
$$

The right premiss of Nes is derivable in a similar way.

$$(*) \; b \subseteq a, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$$

Properties of **G3V**

Basic structural properties

- Weakening and contraction are height-preserving admissible;
- All the rules are height-preserving invertible;
- The cut rule is admissible (syntactic cut elimination).

Soundness

The rules of **G3V** are sound with respect to neighbourhood models for $\mathbb{V}$.

Completeness

We obtain completeness by simulating within **G3V** the internal sequent calculus.

Blocks

- A *block* is a pair consisting of a multiset Σ of formulas and a single formula B, written $[\Sigma \lhd B]$.
- Blocks denote disjunctions of $\preccurlyeq$-formulas:

$$[A_1, \ldots, A_m \lhd B]$$

$$(A_1 \preccurlyeq B) \vee (A_2 \preccurlyeq B) \vee \cdots \vee (A_m \preccurlyeq B)$$

Sequents

- Blocks can occur only in the succedent of a sequent.
- The formula interpretation of a sequent is given by:

$$\Gamma \Rightarrow \Delta', [\Sigma_1 \lhd B_1], \ldots, [\Sigma_n \lhd B_n] \;:=\; \bigwedge \Gamma \rightarrow \bigvee \Delta' \vee \bigvee_{1 \leq i \leq n} \bigvee_{A \in \Sigma_i} (A \preccurlyeq B_i)$$

The calculus $\mathcal{I}^i_{\mathbb{V}}$

Rules of $\mathcal{I}^i_{\mathbb{V}}$

Initial sequents $\quad \Gamma, \bot \Rightarrow \Delta \quad \Gamma, p \Rightarrow \Delta, p$

Propositional rules (standard)

Rules for comparative plausibility

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \preccurlyeq^i_R$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]} \preccurlyeq^i_L$$

Rules for the blocks

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]} \; com^i$$

$$\frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \; jump$$

Example

Derivation of $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

$$
\dfrac{
\dfrac{\dfrac{(AX)}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ jump
\qquad
\dfrac{\dfrac{(AX)}{A \Rightarrow A, B}}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ jump
}{
\dfrac{\dfrac{\dfrac{\Rightarrow [A \lhd B], [B \lhd A]}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow (A \preccurlyeq B) \vee (B \preccurlyeq A)}\ \vee_R
}\ com^i
$$

Properties of $\mathcal{I}^i_{\mathbb{V}}$ Girlando, Lellmann, Olivetti, Pozzato (2016)

Cut elimination

Proved for an equivalent version of the calculus, non-invertible and with contraction rules explicitly defined.

Soundness

If a formula is derivable in the calculus $\mathcal{I}^i_{\mathbb{V}}$, then it is valid in $\mathbb{V}$.

Completeness

If formula A is valid in $\mathbb{V}$, then sequent $\Rightarrow A$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$.

Translation: from $\mathcal{I}^{i}_{\mathbb{V}}$ to **G3V**

Translation

Idea

- $x \Vdash A \preccurlyeq B \;\equiv\; \forall \alpha \in I(x)(\alpha \Vdash^{\exists} B \textit{ implies } \alpha \Vdash^{\exists} A)$
- $[A_1, \ldots, A_n \lhd B] = (A_1 \preccurlyeq B) \vee (A_2 \preccurlyeq B) \vee \cdots \vee (A_n \preccurlyeq B)$
- Each block is interpreted in the language of the labelled sequent calculus as expressing the semantic condition corresponding to the disjunction of $\preccurlyeq$-formulas:

  $$x \Vdash [A_1, \ldots, A_n \lhd B] \;\; \textit{iff} \;\; \forall \alpha \in I(x)(\alpha \Vdash^{\exists} B \textit{ implies } \alpha \Vdash^{\exists} (A_1 \vee \cdots \vee A_n))$$

- We introduce a new neighbourhood label $a \in I(x)$ for each block.

Definition

From $\mathcal{I}^i_{\mathbb{V}}$ to **G3V**

Given a world label x, countably many neighbourhood labels $a, b, c...$ and a multiset $\Sigma = F_1, \ldots, F_k$, define:

- $\Sigma^{t_x} = x : F_1, \ldots, x : F_k$
- $(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd B_1], \ldots, [\Sigma_n \lhd B_n])^{t_x} \ =$

  $a_1, \ldots, a_n \in I(x), a_1 \Vdash^{\exists} B_1, \ldots, a_n \Vdash^{\exists} B_n, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a_1 \Vdash^{\exists} \Sigma_1, \ldots, a_n \Vdash^{\exists} \Sigma_n$

  where for $\Sigma_i = S_i^1, \ldots, S_i^k$

$$a_i \Vdash^{\exists} \Sigma_i \ = \ a_i \Vdash^{\exists} S_i^1, \ldots, a_i \Vdash^{\exists} S_i^k$$

Translation

Theorem

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$, its translation $(\Gamma \Rightarrow \Delta)^{t_x}$ is derivable in **G3V**.

Proof

Induction on the height of the derivation of a sequent $\Gamma \Rightarrow \Delta$, and distinction of cases.

Lemma

Rule $Mon\exists$ is admissible in **G3V**:

$$\frac{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A, b \Vdash^{\exists} A}{b \subseteq a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A} \; Mon\exists$$

- idea: if $\alpha \Vdash^{\exists} A$ is *false*, for all neighbourhood $\beta \subseteq \alpha$ formula $\beta \Vdash^{\exists} A$ is false;
- directly implements rule com^{i} in **G3V**.

Proof: rule com^i

$$\dfrac{\overset{(1)}{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B]} \quad \overset{(2)}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ com^i$$

$$\mathcal{D}_1 \left\{ \dfrac{\dfrac{\overset{(1)^{t_x}}{a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, a \Vdash^{\exists} \Sigma_2, b \Vdash^{\exists} \Sigma_2}}{a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, a \Vdash^{\exists} \Sigma_2, b \Vdash^{\exists} \Sigma_2}\ Wk}{a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2}\ Mon\exists \right.$$

$$\mathcal{D}_2 \left\{ \dfrac{\dfrac{\overset{(2)^{t_x}}{a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2}}{b \subseteq a, a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2}\ Wk}{b \subseteq a, a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2}\ Mon\exists \right.$$

$$\dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{a \in I(x), b \in I(x), a \Vdash^{\exists} A, b \Vdash^{\exists} B, \Gamma^{t_x} \Rightarrow \Delta^{t_x}, a \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_1, b \Vdash^{\exists} \Sigma_2}\ Nes$$

Inverse translation: from **G3V** to $\mathcal{I}^{i}_{\mathbb{V}}$

Inverse translation

Aim

- Prove the converse: if $(\Gamma \Rightarrow \Delta)^{t_x}$ is derivable in **G3V**, then $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$.
- Difficulty: there are **G3V** sequents which are derivable, but that cannot be translated; **G3V** derivable sequents are *more* than $\mathcal{I}^i_{\mathbb{V}}$ derivable sequents.

Proof strategy

- Definition of the inverse translation int_x;
- Definition of *normal form* derivations;
- Theorem: if $\Gamma \Rightarrow \Delta$ is derivable in **G3V** and it can be translated, then $(\Gamma \Rightarrow \Delta)^{int_x}$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$.

Definition

From **G3V** to $\mathcal{I}^i_{\mathbb{V}}$ (1)

We translate only sequents of this form:

$$\mathcal{R}^{\subseteq}, a_1, ..., a_n \in I(x), a_1 \Vdash^{\exists} A_1, ..., a_n \Vdash^{\exists} A_n, x : \Gamma \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n$$

where:

- $\mathcal{R}^{\subseteq}$ contains zero or more inclusions;
- for each $a_i \in I(x)$ there is exactly one formula $a_i \Vdash^{\exists} A_i$ occurring in the antecedent;
- for each $a_i \in I(x)$ there is at least one formula $a_i \Vdash^{\exists} B_i$ in the consequent (but there could be more);
- $a_i \Vdash^{\exists} \Sigma_i \ = \ a_i \Vdash^{\exists} S_i^1, \ldots, a_i \Vdash^{\exists} S_i^k$
- Γ, Δ contain only propositional and $\preccurlyeq$ formulas.

Definition

From **G3V** to $\mathcal{I}^{i}_{\mathbb{V}}$ (2)

$$(\mathcal{R}^{\subseteq}, a_1, ..., a_n \in I(x), a_1 \Vdash^{\exists} A_1, ..., a_n \Vdash^{\exists} A_n, x : \Gamma \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, ..., a_n \Vdash^{\exists} \Sigma_n)^{int_x}$$

$$:= \Gamma \Rightarrow \Delta, \Pi$$

where:

- Γ is obtained from $x : \Gamma$ by removing the label x;
- Δ is obtained from $x : \Delta$ by removing the label x;
- Π contains n blocks

$$[\Lambda_1 \lhd A_1], ..., [\Lambda_n \lhd A_n]$$

and

$$\Lambda_i = \Sigma_i \cup \bigcup \{\Sigma_j \mid a_i \subseteq a_j \textit{ occurs in the antecedent}\}.$$

Definition

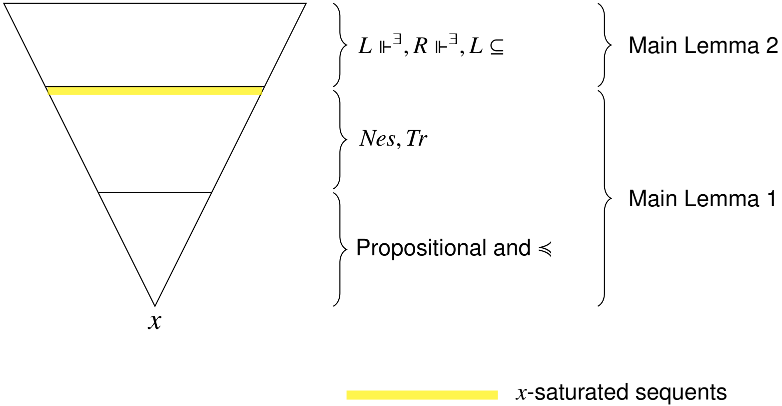
Example

$$(a_1 \subseteq a_2, a_1 \in I(x), a_2 \in I(x), a_3 \in I(x), a_1 \Vdash^{\exists} A_1, a_2 \Vdash^{\exists} A_2, a_3 \Vdash^{\exists} A_3, x : \Gamma \Rightarrow \\ \Rightarrow x : \Delta, a_1 \Vdash^{\exists} \Sigma_1, a_2 \Vdash^{\exists} \Sigma_2, a_3 \Vdash^{\exists} \Sigma_3)^{int_x}$$

$$=$$

$$\Gamma^{int_x} \Rightarrow \Delta^{int_x}, [\Sigma_1, \Sigma_2 \lhd A_1], [\Sigma_2 \lhd A_2], [\Sigma_3 \lhd A_3]$$

Normal form derivations



Almost there..

Main Lemma 1

Let $\Gamma \Rightarrow \Delta$ be a derivable **G3V** sequent that can be translated in $\mathcal{I}^i_{\mathbb{V}}$; let x be a world label occurring in it. The sequent is derivable with a derivation in normal form with respect to x and from x-saturated sequents $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$. Moreover, it holds that $(\Gamma \Rightarrow \Delta)^{int_x}$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$ from $(\Gamma_1 \Rightarrow \Delta_1)^{int_x}, \ldots, (\Gamma_n \Rightarrow \Delta_n)^{int_x}$.

Main Lemma 2

Let $\Gamma_i \Rightarrow \Delta_i$ be one of $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$. If $\Gamma_i \Rightarrow \Delta_i$ is not an initial sequent, it is derivable in **G3V** from a sequent $\Gamma'_i \Rightarrow \Delta'_i$, such that in $\mathcal{I}^i_{\mathbb{V}}$ $(\Gamma_i \Rightarrow \Delta_i)^{int_x}$ is obtained from $(\Gamma'_i \Rightarrow \Delta'_i)^{int_x}$ by *jump*.

Proof of Main lemma 2

Jump lemma

Let $\Gamma \Rightarrow \Delta$ be a derivable **G3V** sequent. For each label x occurring in it, it holds that either:

1) $\Gamma_x^\Gamma \Rightarrow \Delta_x^\Gamma$ is derivable in **G3V** or
2) $\Gamma - \Gamma_x^\Gamma \Rightarrow \Delta - \Delta_x^\Gamma$ is derivable in **G3V**.

where Γ_x^Γ contains formulas labelled with x or with labels generated (transitively) from x.

Example

$$y \in a, y \in b, z \in a, y : B, z : A \Rightarrow y : A, y : B, z : B, a \Vdash^\exists A, b \Vdash^\exists B$$

- either $y : B \Rightarrow y : A, y : B$ is derivable, or
- $y \in a, y \in b, z \in a, z : A \Rightarrow z : B, a \Vdash^\exists A, b \Vdash^\exists B$ is derivable.

Finally

Theorem

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in **G3V** and it can be translated, there is a derivation of $(\Gamma \Rightarrow \Delta)^{int_x}$ in $\mathcal{I}^i_{\mathbb{V}}$.

Proof

By induction on the modal degree of a **G3V** sequent (level of nesting of $\preccurlyeq$). The modal degree of each $\Gamma'_i \Rightarrow \Delta'_i$ is lesser than the modal degree of $\Gamma \Rightarrow \Delta$; thus by IH the sequent is derivable, and a derivation of it can be transformed into a derivation of its translation in $\mathcal{I}^i_{\mathbb{V}}$.

Example

Derivation of $A \preccurlyeq B \vee B \preccurlyeq A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(AX)}{a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B, y : B}}{a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ R\Vdash^{\exists}}{a \subseteq b, a \in I(x), b \in I(x), y \in a, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ L\subseteq}{a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ L\Vdash^{\exists} \qquad \vdots}{a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B}\ Nes}{a \in I(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A}\ R\preccurlyeq}{\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A}\ R\preccurlyeq}{\Rightarrow x : A \preccurlyeq B \vee x : B \preccurlyeq A}\ L\vee
$$

Example

Up to Nes

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow [A, B \lhd B], [B \lhd A] \qquad \vdots}{\Rightarrow [A \lhd B], [B \lhd A]}\ com^i}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B \vee B \preccurlyeq A}\ L\vee$$

Example

Main Lemma 2

- $y : B \Rightarrow y : A, y : B$ is derivable
- $(y : B \Rightarrow y : A, y : B)^{int_y} = B \Rightarrow A, B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, B}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ jump \qquad \vdots}{\Rightarrow [A \lhd B], [B \lhd A]}\ com^i}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B \lor B \preccurlyeq A}\ L\lor}{}
$$

Example

Main Lemma 2

- $y : B \Rightarrow y : A, y : B$ is derivable
- $(y : B \Rightarrow y : A, y : B)^{int_y} = B \Rightarrow A, B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, B}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ jump \qquad \dfrac{A, B \Rightarrow A}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ jump}{\Rightarrow [A \lhd B], [B \lhd A]}\ com^i}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \preccurlyeq^i_R}{\Rightarrow A \preccurlyeq B \vee B \preccurlyeq A}\ L\vee
$$

Conclusions

To sum up

Theorem (from $\mathcal{I}^i_{\mathbb{V}}$ to G3V)

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$, its translation $(\Gamma \Rightarrow \Delta)^{t_x}$ is derivable in **G3V**, and we can construct a **G3V** derivation from the $\mathcal{I}^i_{\mathbb{V}}$ derivation for it.

Theorem (from G3V to $\mathcal{I}^i_{\mathbb{V}}$)

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in **G3V** and it can be translated, there is a derivation of $(\Gamma \Rightarrow \Delta)^{int_x}$ in $\mathcal{I}^i_{\mathbb{V}}$, and we can construct a $\mathcal{I}^i_{\mathbb{V}}$ derivation from the **G3V** derivation for it.

Side result

Completenes of $\mathcal{I}^i_{\mathbb{V}}$

If formula A is valid in $\mathbb{V}$, then sequent $\Rightarrow A$ is derivable in $\mathcal{I}^i_{\mathbb{V}}$.

Completeness of **G3V** (Corollary)

Since the rules of $\mathcal{I}^i_{\mathbb{V}}$ can be simulated in **G3V**, **G3V** is complete with respect to $\mathbb{V}$.

Conclusions

Internal vs external sequent calculi

The mapping allowed us to have some insight on both calculi, external and internal:

- The mapping makes explicit the semantic intuition “hidden” in the rules of the internal sequent calculus;
- The mapping hints that only a part of the information contained in a labelled sequent is relevant (compare with $jump$).

Future work

- Extend the proof to extensions of $\mathbb{V}$;
- Apply the $Jump$ lemma to other labelled calculi.

Thank you!

Thank you!

Questions?

Example

Derivation of $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

$$
\dfrac{
\dfrac{\overset{(AX)}{B \Rightarrow A, B}}{\Rightarrow [A, B \lhd B], [B \lhd A]}\ jump
\qquad
\dfrac{\overset{(AX)}{A \Rightarrow A, B}}{\Rightarrow [A \lhd B], [A, B \lhd A]}\ jump
}{
\dfrac{
\dfrac{
\dfrac{\Rightarrow [A \lhd B], [B \lhd A]}{\Rightarrow [A \lhd B], B \preccurlyeq A}\ \preccurlyeq^i_R
}{\Rightarrow A \preccurlyeq B, B \preccurlyeq A}\ \preccurlyeq^i_R
}{\Rightarrow (A \preccurlyeq B) \vee (B \preccurlyeq A)}\ \vee_R
}\ com^i
$$

Example

$$
\begin{array}{cl}
(AX) & \\
y : B \Rightarrow y : A, y : B & \\
\hline
a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B, y : A, y : B & Wk \\
\hline
a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B & R\Vdash^{\exists} \\
\hline
a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B & L\Vdash^{\exists} \\
\hline
a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B & L\Vdash^{\exists} \\
\hline
a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B \ldots & Mon\exists \\
\hline
a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B & N \\
\hline
a \in I(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A & R\preccurlyeq \\
\hline
\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A & R\preccurlyeq
\end{array}
$$

Example

$$\frac{(AX)}{y : B \Rightarrow y : A, y : B}$$

$$Wk$$

$$a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B, y : A, y : B$$

$$R\Vdash^{\exists}$$

$$a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$$

$$L\Vdash^{\exists}$$

$$a \subseteq b, a \in I(x), b \in I(x), y \in a, y \in b, y : B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$$

$$L\Vdash^{\exists}$$

$$a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, a \Vdash^{\exists} B, b \Vdash^{\exists} B$$

$$Mon\exists$$

$$a \subseteq b, a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$$

$$N$$

$$a \in I(x), b \in I(x), a \Vdash^{\exists} B, b \Vdash^{\exists} A \Rightarrow a \Vdash^{\exists} A, b \Vdash^{\exists} B$$

$$R\preccurlyeq$$

$$a \in I(x), a \Vdash^{\exists} B \Rightarrow x : B \preccurlyeq A, a \Vdash^{\exists} A$$

$$R\preccurlyeq$$

$$\Rightarrow x : A \preccurlyeq B, x : B \preccurlyeq A$$

Definitions (3)

Normal form derivation

Given a sequent $\Gamma \Rightarrow \Delta$ and a label x, a derivation is in *normal form with respect to x* if it is built applying the rules bottom-up in the following order:

- propositional and $\preccurlyeq$ rules applied to formulas $x : A$;
- Nes, preceded (if possible) by Tr applied to formulas $a \in I(x)$;
- finally, rules $L \Vdash^{\exists}$, $R \Vdash^{\exists}$ and $L \subseteq$.

x-saturated sequent

A sequent is *saturated with respect to* x if propositional and $\preccurlyeq$ rules have been applied to all formulas labelled with x, and if the sequent is saturated with respect to nesting and transitivity.

Proof of Main Lemma 1

Induction on the saturation degree; by distinction of cases.

Case of $(R \preccurlyeq)$

$$\frac{a \Vdash^{\exists} B, a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : A \preccurlyeq B} \; R{\preccurlyeq} \; (a \text{ new})$$

$$(a \Vdash^{\exists} B, a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A)^{int_x} = \Gamma^{int_x} \Rightarrow \Delta^{int_x}, [A \lhd B]$$

$$\frac{\Gamma^{int_x} \Rightarrow \Delta^{int_x}, [A \lhd B]}{\Gamma^{int_x} \Rightarrow \Delta^{int_x}, A \preccurlyeq B} \; \preccurlyeq^{i}_{R}$$