Multi-contextual structures and Label-free calculi for Intuitionistic Modal Logics

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Plan

Modal Logics

- Modelling of systems : how to capture specific aspects like temporality, spatiality, resource management, etc...
  $\Rightarrow$ Modalities

- Various kinds of modal logics : classical, intuitionistic, fuzzy, linear.

- Extension of classical logic with modalities :
  $\Box$ (necessity) and $\Diamond$ (possibility)
  - modalities interpreted in a set of worlds with an accessibility relation.
  - modal logics differ by the properties associated to the accessibility relation : reflexivity (T), symmetry (B), transitivity (4), euclidness (5).

Intuitionistic Modal Logics

- Intuitionistic reasoning in modal logics (Simpson 94)
- Algorithmic contents of proofs (Curry-Howard isomorphism)
- Applications in computer science :
  - Formal verification of hardware (Fairtlough et al. 94)
  - Definition of programming languages (Davies et al. 01, Murphy VII et al. 04)
  - Expressivity of properties in communicating systems (Stirling 87)

Proof theory in modal logics

Existing works and results

- Natural deduction systems in classical case but rare in intuitionistic case : problem to deal with $\diamondsuit$ (Simpson 94).
- Natural deduction systems satisfying normalization are based on labels.
  - Labels explicitly integrate semantic information like the accessibility relation
- Sequent calculi with labels for various modal logics (Negri 05) but without some properties like subformula property.

Problem : how to design label-free calculi with good properties for intuitionistic modal logics ?

Our approach

Definition of multi-contextual structures without labels

- Such a structure for sequent calculi in classical modal logics : deep sequent (Brünnler 09)
- No similar structure for intuitionistic modal logics with natural deduction and sequent formalisms.
- Preliminary results in IS5 : MC-sequent (Galmiche-Salhi 10)

Design of label-free calculi for intuitionistic modal logics

- Natural deduction and sequent calculi systems
- Intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5
- Good properties : normalization, cut-elimination, subformula properties.
- Decision procedures and syntactic proofs of decidability in some cases.

Results

- Definition of a new multi-contextual (sequent) structure : T-sequent
- Label-free proof systems for the intuitionistic logic IK with normalization/cut-elimination property and subformula property.
- Label-free proof systems for intuitionistic modal logics obtained from the combinations of the axioms T, B, 4 and 5
  - Normalization/cut-elimination property
  - Subformula property
- Decision procedures for some intuitionistic modal logics

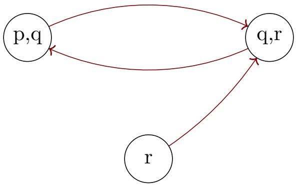
Plan

Classical modal logics (normal)

- Extensions of classical logic with modalities : $\Box$, $\Diamond$

$$A ::= p \mid \bot \mid A \land A \mid A \lor A \mid A \supset A \mid \Box A \mid \Diamond A$$

- Kripke semantics : Models $\mathcal{M} = (W, R, V)$ with W set of worlds and R relation of accessibility.



Classical modal logics (normal)

- Extensions of classical logic with modalities : $\Box$, $\Diamond$

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A$$

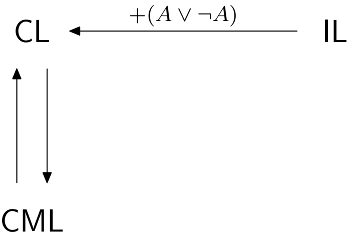
- Kripke semantics : Models $\mathcal{M} = (W, R, V)$ with W set of worlds and R relation of accessibility.
- Satisfaction relation :
  - $w \vDash_{\mathcal{M}} p$ iff $p \in V(w)$
  - $w \vDash_{\mathcal{M}} \bot$ never
  - $w \vDash_{\mathcal{M}} A \wedge B$ iff $w \vDash_{\mathcal{M}} A$ and $w \vDash_{\mathcal{M}} B$
  - $w \vDash_{\mathcal{M}} A \vee B$ iff $w \vDash_{\mathcal{M}} A$ or $w \vDash_{\mathcal{M}} B$
  - $w \vDash_{\mathcal{M}} A \supset B$ iff if $w \vDash_{\mathcal{M}} A$ then $w \vDash_{\mathcal{M}} B$
  - $w \vDash_{\mathcal{M}} \Box A$ iff for all $w' \in W$, if $R(w, w')$ then $w' \vDash_{\mathcal{M}} A$
  - $w \vDash_{\mathcal{M}} \Diamond A$ iff if there exists $w' \in W$ such that $R(w, w')$ and $w' \vDash_{\mathcal{M}} A$

Classical modal logics (normal)

- Classical modal models define validity in the minimal modal logic K : a formula A is valid in K iff A is valid in all classical modal models (Chellas 80)
- Other modal logics built from combinations of the axioms T, B, 4 and 5 are defined by classes of classical modal models.
- Each axiom corresponds to a property of the accessibility relation in each model :
  - (T) Reflexivity : $\forall w.R(w,w)$;
  - (B) Symmetry : $\forall w,w'.R(w,w') \supset R(w',w)$;
  - (4) Transitivity : $\forall w,w',w''.(R(w,w') \wedge R(w',w'')) \supset R(w,w'')$;
  - (5) Euclidness : $\forall w,w',w''.(R(w,w') \wedge R(w,w'')) \supset R(w',w'')$.
- For Th $\subseteq \{T,B,4,5\}$ the class of models defining the logics KTh, corresponds to models in which the accessibility relations satisfy the given properties

Intuitionistic Modal Logics

- Simpson's approach (Simpson 94) :



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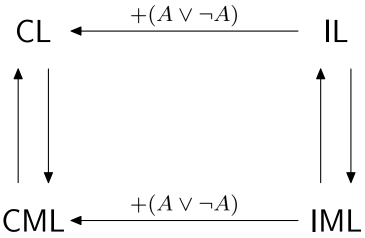

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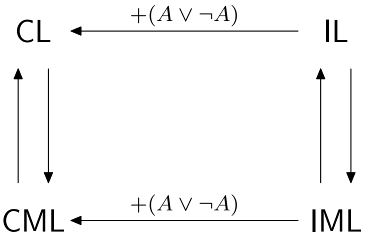

- Classical modal logics (normal) minus $A \vee \neg A$.

Intuitionistic Modal Logics

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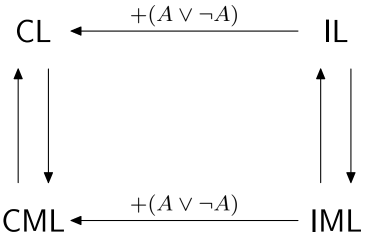

- Classical modal logics (normal) minus $A \vee \neg A$.
- Relationships between IML and CML like the ones between IL and CL.

Intuitionistic Modal Logics

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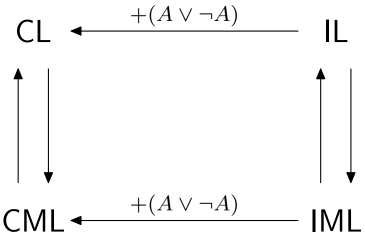

- Classical modal logics (normal) minus $A \vee \neg A$.
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- Modalities are independent.

Intuitionistic Modal Logics

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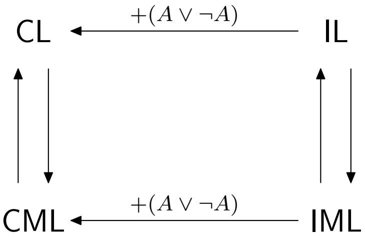


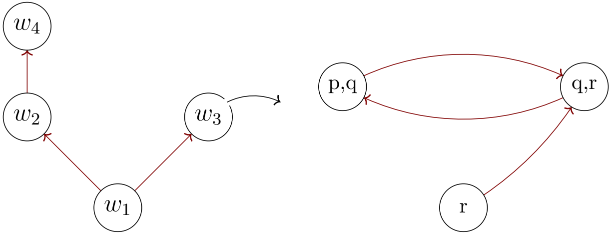
- Classical modal logics (normal) minus $A \vee \neg A$.
- Relationships between IML and CML like the ones between IL and CL.
- Modalities are independent.
- First intuitionistic modal logics (Fitch 48,Prior 57)

Intuitionistic modal logics (normal)

- Intuitionistic reasoning associated to modal logics

⇒ Classical modal logics minus $A \vee \neg A$.

- Kripke semantics : Models $\mathcal{M} = (W, \leqslant, D_{w \in W}, R_{w \in W}, V_{w \in W})$ with W set of worlds and D_w set of modal worlds.



  - $w \leqslant w'$ entails $D_w \subseteq D_{w'}$ & $R_w \subseteq R_{w'}$ & $V_w \subseteq V_{w'}$

Intuitionistic modal logics (normal)

- Satisfaction relation :
  - $w, d \vDash_{\mathcal{M}} p$ iff $p \in V_w(d)$
  - $w, d \vDash_{\mathcal{M}} \bot$ never
  - $w, d \vDash_{\mathcal{M}} A \wedge B$ iff $w, d \vDash_{\mathcal{M}} A$ and $w, d \vDash_{\mathcal{M}} B$
  - $w, d \vDash_{\mathcal{M}} A \vee B$ iff $w, d \vDash_{\mathcal{M}} A$ or $w, d \vDash_{\mathcal{M}} B$
  - $w, d \vDash_{\mathcal{M}} A \supset B$ iff for all $w' \geqslant w$, if $w', d \vDash_{\mathcal{M}} A$ then $w', d \vDash_{\mathcal{M}} B$
  - $w, d \vDash_{\mathcal{M}} \Box A$ iff for all $w' \geqslant w$ and for all $d' \in D_{w'}$, if $R_{w'}(d, d')$ then $w', d' \vDash_{\mathcal{M}} A$
  - $w, d \vDash_{\mathcal{M}} \Diamond A$ iff there exists $d' \in D_w$ such that $R_w(d, d')$ and $w, d \vDash_{\mathcal{M}} A$
- Monotonicity : $w \leqslant w'$ and $w, d \vDash A$ entails $w', d \vDash A$
- For any Th $\subseteq \{T, B, 4, 5\}$ we call IKTh the intuitionistic version of the classical modal logic KTh.

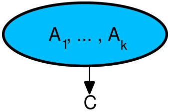
Intuitionistic modal logics and proof systems

- A Hilbert axiomatic system for IK (Simpson 94).
- Natural deduction system for S4 and S5 and their intuitionistic versions (Prawitz 65, Bierman-de Paiva 00)
- Proof systems for intuitionistic modal logic are rare
  - Labelled calculi (Dosen 86, Simpson 94) but no subformula property.
  - Label-free sequent calculi for classical modal logics (Brünnler 09).
  - Label-free systems for fragments without $\Diamond$ of IK, IS4 and IS5 (Ono 77, Bierman et De Paiva 2000) but cut-elimination property.
- **Multi-contextual structures and disjunctive property**

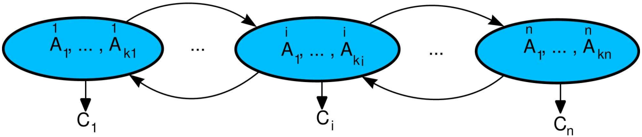
Plan

Multi-contextual structures

- Sequent : $A_1, \ldots, A_k \vdash C$
  ($A_1 \wedge \cdots \wedge A_k \supset C$)



- Hypersequent : $\Gamma_1 \vdash C_1 \mid \ldots \mid \Gamma_n \vdash C_n$
  ($(\bigwedge \Gamma_1 \supset C_1) \vee \cdots \vee (\bigwedge \Gamma_n \supset C_n)$)



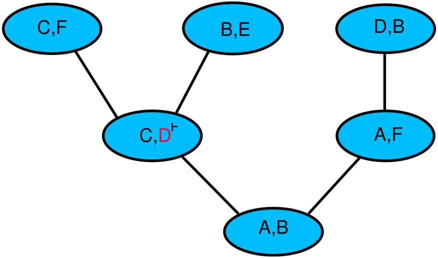
- Deep sequent : $A_1, \ldots, A_k, [\Gamma_1], \ldots, [\Gamma_l]$ with $\{A_1, ..A_k\}$ multiset of formulas and $\{\Gamma_1, ..\Gamma_l\}$ multiset of deep sequents.
  ($\bigvee A_i \vee \Box\mathcal{F}(\Gamma_1) \vee \ldots \vee \Box\mathcal{F}(\Gamma_l)$)

Tree-sequent (or T-sequent)

- T-context : $A_1, \ldots, A_k, \langle\Gamma_1\rangle, \ldots, \langle\Gamma_l\rangle$ with $\{A_1, ..A_k\}$ multiset of formulas and $\{\Gamma_1, ..\Gamma_l\}$ multiset of T-contexts.
  $(\bigwedge A_i \wedge \Diamond\mathcal{F}(\Gamma_1) \wedge \ldots \wedge \Diamond\mathcal{F}(\Gamma_k))$

- T-sequent :
  - $\Gamma, C^{\vdash}$ with Γ is a T-context
    $(\mathcal{F}(\Gamma) \supset C)$.
  - $\Gamma, \langle\mathcal{S}\rangle$ with Γ is a T-context and $\mathcal{S}$ is a T-sequent
    $(\mathcal{F}(\Gamma) \supset \Box(\mathcal{F}(\mathcal{S})))$

T-sequent



$$A, B, \langle C, D^{\vdash}, \langle C, F \rangle, \langle B, E \rangle \rangle, \langle A, F, \langle D, B \rangle \rangle$$

T-sequent : a new multi-contextual structure

A T-sequent is different from a deep sequent (Brünnler 2009)

- one distinguishes one formula (the marked one)
- one deals with the two modalities $\Box, \Diamond$ and not only with $\Box$.

nT-contexts and inference rules

- nT-context : a T-context with n occurrences of $\{\}$ (hole).
  - Notation : $\Gamma \overbrace{\{\} \cdots \{\}}^{n\ fois}$
  - Hole substitution : $\Gamma\{\Delta_1\} \cdots \{\Delta_n\}$
  - Example :

$$\Gamma\{\} = \Box(A \supset B), \Diamond A, \langle A, \{\} \rangle$$
$$\Gamma\{C, D^{\vdash}\} = \Box(A \supset B), \Diamond A, \langle A, C, D^{\vdash} \rangle$$

- Form of inference rules :

$$\frac{\Gamma\{\Delta_1^1\} \cdots \{\Delta_k^1\} \quad \cdots \quad \Gamma\{\Delta_1^l\} \cdots \{\Delta_k^l\}}{\Gamma\{\Delta_1\} \cdots \{\Delta_k\}} \ [R]$$

Plan

The sequent calculus G_{IK} (1)

- Left rules : they deal with the T-context
- Right rules : they deal with the marked formulae
- Propositional rules :

$$\frac{}{\Gamma\{A, A^{\vdash}\}}\ [id] \qquad \frac{}{\Gamma\{\bot\}\{C^{\vdash}\}}\ [\bot_L]$$

$$\frac{\Gamma\{A_1 \wedge A_2, A_i\}\{C^{\vdash}\}}{\Gamma\{A_1 \wedge A_2\}\{C^{\vdash}\}}\ [\wedge_L^i] \qquad \frac{\Gamma\{A^{\vdash}\} \qquad \Gamma\{B^{\vdash}\}}{\Gamma\{A \wedge B^{\vdash}\}}\ [\wedge_R]$$

$$\frac{\Gamma\{A \vee B, A\}\{C^{\vdash}\} \qquad \Gamma\{A \vee B, B\}\{C^{\vdash}\}}{\Gamma\{A \vee B\}\{C^{\vdash}\}}\ [\vee_L] \qquad \frac{\Gamma\{A_i^{\vdash}\}}{\Gamma\{A_1 \vee A_2^{\vdash}\}}\ [\vee_R^i]$$

$$\frac{\Gamma\{A \supset B, A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A \supset B, B\}\{C^{\vdash}\}}{\Gamma\{A \supset B\}\{C^{\vdash}\}}\ [\supset_L] \qquad \frac{\Gamma\{A, B^{\vdash}\}}{\Gamma\{A \supset B^{\vdash}\}}\ [\supset_R]$$

$$\frac{\Gamma\{A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [Cut]$$

The sequent calculus G_{IK} (2)

- Modal rules :

$$\frac{\Gamma\{\langle A\rangle, \Diamond A\}\{C^{\vdash}\}}{\Gamma\{\Diamond A\}\{C^{\vdash}\}}\ [\Diamond_L] \qquad \frac{\Gamma\{\langle \Delta, A^{\vdash}\rangle\}}{\Gamma\{\langle \Delta\rangle, \Diamond A^{\vdash}\}}\ [\Diamond_R]$$

$$\frac{\Gamma\{\langle \Delta, A\rangle, \Box A\}}{\Gamma\{\langle \Delta\rangle, \Box A\}}\ [\Box_L] \qquad \frac{\Gamma\{\langle A^{\vdash}\rangle\}}{\Gamma\{\Box A^{\vdash}\}}\ [\Box_R]$$

$$\frac{\Gamma\{A^{\vdash}\}\{\emptyset\} \qquad \Gamma\{A\}\{C^{\vdash}\}}{\Gamma\{\emptyset\}\{C^{\vdash}\}}\ [Cut]$$

Construction of calculi G_{IKTh} (1)

We associate to each logic IKTh, with Th $\subseteq \{T, B, 4, 5\}$, the sequent calculus G_{IKTh} that is obtained from the previous rules as follows :

- if IKTh is IS5 then G_{IKTh} is obtained from G_{IK} by replacing the rules $[\Box_L]$ and $[\Diamond_R]$ by the rules $[\Box_L^{\mathrm{IS5}}]$ and $[\Diamond_R^{\mathrm{IS5}}]$;
- if IKTh is IB4 then G_{IKTh} is obtained from G_{IK} by replacing the rules $[\Box_L]$ and $[\Diamond_R]$ by the rules $[\Box_L^{\mathrm{IB4}}]$ and $[\Diamond_R^{\mathrm{IB4}}]$;
- otherwise G_{IKTh} is obtained by adding to G_{IK} the rules $[\Box_L^x]$ and $[\Diamond_R^x]$ for all $x \in$ Th.

Construction of calculi G_{IKTh} (2)

$$\frac{\Gamma\{\Box A, A\}}{\Gamma\{\Box A\}}\ [\Box_L^T] \qquad \frac{\Gamma\{A^\vdash\}}{\Gamma\{\Diamond A^\vdash\}}\ [\Diamond_R^T] \qquad \frac{\Gamma\{\langle\Delta, \Box A\rangle, A\}}{\Gamma\{\langle\Delta, \Box A\rangle\}}\ [\Box_L^B] \qquad \frac{\Gamma\{\langle\Delta\rangle, A^\vdash\}}{\Gamma\{\langle\Delta, \Diamond A^\vdash\rangle\}}\ [\Diamond_R^B]$$

$$\frac{\Gamma\{\Delta\{A\}, \Box A\}}{\Gamma\{\Delta\{\emptyset\}, \Box A\}}\ [\Box_L^4](depth(\Delta\{\}) > 1) \qquad \frac{\Gamma\{\Delta\{A^\vdash\}\}}{\Gamma\{\Delta\{\emptyset\}, \Diamond A^\vdash\}}\ [\Diamond_R^4](depth(\Delta\{\}) > 1)$$

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^5](depth(\Gamma\{\}\{\emptyset\}) > 0 \ \texttt{and}\ depth(\Gamma\{\emptyset\}\{\}) > 0)$$

$$\frac{\Gamma\{\emptyset\}\{A^\vdash\}}{\Gamma\{\Diamond A^\vdash\}\{\emptyset\}}\ [\Diamond_R^5](depth(\Gamma\{\}\{\emptyset\}) > 0 \ \texttt{and}\ depth(\Gamma\{\emptyset\}\{\}) > 0)$$

The depth of a $1T$-context $\Gamma\{\}$ is defined as follows : $depth(\Gamma, \{\}) = 0$; $depth(\Gamma, \langle\Delta\{\}\rangle) = 1 + depth(\Delta\{\})$.

Construction of calculi G_{IKTh} (3)

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^{\mathrm{IB4}}](sp(\Gamma\{\Box A^{\vdash}\}\{\emptyset\})) \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_R^{\mathrm{IB4}}](sp(\Gamma\{\emptyset\}\{A^{\vdash}\}))$$

$$\frac{\Gamma\{\Box A\}\{A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ [\Box_L^{\mathrm{IS5}}] \qquad \frac{\Gamma\{\emptyset\}\{A^{\vdash}\}}{\Gamma\{\Diamond A^{\vdash}\}\{\emptyset\}}\ [\Diamond_R^{\mathrm{IS5}}]$$

Let $\mathcal{S}$ be a T-sequent, $sp(\mathcal{S})$ is true iff if the depth of the tree corresponding to $\mathcal{S}$ is greater than 0.

Theorem [Soundness]
If a T-sequent has a preuve in G_{IKTh} then it is valid in IKTh.

Theorem [Completeness]
If a T-sequent is valid in IKTh then it has a proof in G_{IKTh}.

G_{IKTh} - an example of proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\langle\Box A\rangle, A, A^{\vdash}}\,[id]}{\langle\Box A\rangle, A^{\vdash}}\,[\Box_L^B]}{\Diamond\Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]
$$

G_{IKTh} - an example of proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\langle \Box A \rangle, A, A^{\vdash}}\,[id]}{\langle \Box A \rangle, A^{\vdash}}\,[\Box_L^B]}{\Diamond \Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond \Box A \supset A^{\vdash}}\,[\supset_R]$$

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G_{IKTh} - an example of proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\langle \Box A \rangle, A, A^{\vdash}}\,[id]}{\langle \Box A \rangle, A^{\vdash}}\,[\Box^{B}_{L}]}{\Diamond \Box A, A^{\vdash}}\,[\Diamond_L]}{\Diamond \Box A \supset A^{\vdash}}\,[\supset_R]
$$

G_{IKTh} - an example of proof

$$
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$$

G_{IKTh} - an example of proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\langle \Box A \rangle, A, A^{\vdash}}\ {\color{red}[id]}}{\langle \Box A \rangle, A^{\vdash}}\ [\Box_L^B]}{\Diamond \Box A, A^{\vdash}}\ [\Diamond_L]}{\Diamond \Box A \supset A^{\vdash}}\ [\supset_R]$$

Properties of G_{IKTh}

Theorem [Cut-elimination property]
If $\mathcal{S}$ is a T-sequent has a proof in G_{IKTh} then there is a proof of $\mathcal{S}$ without the (cut) rule.

Theorem [Subformula property]
If $\mathcal{S}$ is a T-sequent valid in IKTh, then there exists a proof of $\mathcal{S}$ in G^{-}_{IKTh} containing only subformulae of the formulae appearing in $\mathcal{S}$.

Theorem [Depth property]
Let $\mathcal{S}$ be a T-sequent and $\mathcal{D}$ a proof of $\mathcal{S}$ in G^{-}_{IKTh} for Th $\in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$. If $\mathcal{S}'$ is a T-sequent in $\mathcal{D}$ then its depth is less or equal to $d(\mathcal{S}) + nest(\mathcal{S})$.

Plan

T-sequent calculi and decision procedures (1)

A **preorder on T-sequents** :

- Relation $\rightarrow_c$ (contraction) defined by :
$\Gamma\{\Delta, \Delta\} \rightarrow_c \Gamma\{\Delta\}$, $\Gamma\{\langle\Delta, C^{\vdash}\rangle, \langle\Delta\rangle\} \rightarrow_c \Gamma\{\langle\Delta, C^{\vdash}\rangle\}$.
- Relation $\rightarrow_w$ (weakening) defined by :
$\Gamma\{C^{\vdash}\}\{\emptyset\} \rightarrow_w \Gamma\{C^{\vdash}\}\{\Sigma\}$ (Σ is T-context).

Preorder on T-sequents $\mathcal{S} \lesssim \mathcal{S}'$ defined by :
$\mathcal{S} \lesssim \mathcal{S}'$ if and only if $\mathcal{S}(\rightarrow_c + \rightarrow_w)^*\mathcal{S}'$ where $(\rightarrow_c + \rightarrow_w)^*$ is the reflexive and transitive closure of the union of the two relations.
$\mathcal{S} \cong \mathcal{S}'$ if and only if $\mathcal{S} \lesssim \mathcal{S}'$ and $\mathcal{S}' \lesssim \mathcal{S}$.

Proposition

Let $\mathcal{S}$ and $\mathcal{S}'$ be T-sequents such that $\mathcal{S} \lesssim \mathcal{S}'$. If $\vdash^n_{G^-_{\mathrm{IKTh}}} \mathcal{S}$ then $\vdash^n_{G^-_{\mathrm{IKTh}}} \mathcal{S}'$.

T-sequent calculi and decision procedures (2)

A **notion of redundancy** with T-sequents

A derivation is said to be **redundant** if it contains two T-sequents $\mathcal{S}_1$ and $\mathcal{S}_2$, with $\mathcal{S}_1$ occurring strictly above $\mathcal{S}_2$ in the same branch, such that $\mathcal{S}_1 \lesssim \mathcal{S}_2$. It is said to be **irredundant** if it is not redundant

Proposition [Irredundant proof]
For all Th $\subseteq \{T, B, 4, 5\}$, if a T-sequent is valid in IKTh, then it has an irredundant proof in G^{-}_{IKTh}.

Proposition [Finite partition]
Let $\mathcal{S}$ be a T-sequent, Th $\in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$ and $\mathcal{D}$ be a derivation of $\mathcal{S}$ in G^{-}_{IKTh}. The set of all T-sequents appearing in $\mathcal{D}$ is partitioned into a finite set of equivalence classes by $\cong$.

Then the set $\mathcal{B}$ of all branches of all T-sequents appearing in $\mathcal{D}$ is finite.

T-sequent calculi and decision procedures (3)

A decision procedure for IKTh with Th $\in \{\emptyset, \{T\}, \{B\}, \{T, B\}\}$, that is based on G^{-}_{IKTh} calculus and the search of an irredundant proof of the given T-sequent.

Let $\mathcal{S}$ be a T-sequent.
- **Step** 1. We start with the derivation containing only $\mathcal{S}$ which is the unique irredundant derivation of size 1. If this derivation is a proof then we return it. Otherwise we move to the next step.
- **Step** $i + 1$. We build the set of all the irredundant derivations of size $i + 1$. If this set contains a proof of $\mathcal{S}$ then we return it. Otherwise if this set is empty then $\mathcal{S}$ is not valid, else we move to the next step.

Correctness : from Proposition [Irredundant proof] and soundness and completeness of the T-sequent calculus.
Termination : from Proposition [Finite partition] and there is only a finite number of rule applications that extend the size from i to $i + 1$.

T-sequent calculi and decision procedures (4)

An example (in IK$\{T\}$)

Step 1 : $Der_1 = \{\Diamond\Box A \supset A^{\vdash}\}$

T-sequent calculi and decision procedures (4)

An example (in IK$\{T\}$)

Step 1 : $Der_1 = \{\Diamond\Box A \supset A^{\vdash}\}$

Step 2 : $Der_2 = \{\dfrac{\Diamond\Box A, A^{\vdash}}{\Diamond\Box A \supset A^{\vdash}}\,[\supset_R]\}$

T-sequent calculi and decision procedures (4)

An example (in IK$\{T\}$)

Step 1 : $Der_1 = \{\Diamond\Box A \supset A^{\vdash}\}$

Step 2 : $Der_2 = \{\dfrac{\Diamond\Box A, A^{\vdash}}{\Diamond\Box A \supset A^{\vdash}}[\supset_R]\}$

Step3 : $Der_3 = \{ \dfrac{\dfrac{\Diamond\Box A, \langle\Box A\rangle, A^{\vdash}}{\Diamond\Box A, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}[\supset_R] \}$

T-sequent calculi and decision procedures (4)

An example (in IK$\{T\}$)

Step 1 : $Der_1 = \{\Diamond\Box A \supset A^{\vdash}\}$

Step 2 : $Der_2 = \{\dfrac{\Diamond\Box A, A^{\vdash}}{\Diamond\Box A \supset A^{\vdash}}[\supset_R]\}$

Step3 : $Der_3 = \{ \dfrac{\dfrac{\Diamond\Box A, \langle\Box A\rangle, A^{\vdash}}{\Diamond\Box A, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}[\supset_R] \}$

Step 4 : $Der_4 = \{ \dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle\Box A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle\Box A\rangle, A^{\vdash}}[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}[\supset_R] \}$

T-sequent calculi and decision procedures (4)

An example (in IK$\{T\}$)

- The derivations of size 5 that we can obtain from the derivation in Der_4 are :

$$\dfrac{\dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle \Box A, A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}}[\Box_L^T]}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}[\supset_R] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Diamond\Box A, \langle \Box A\rangle, \langle \Box A, A\rangle, A^{\vdash}}{\Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A, \langle \Box A\rangle, A^{\vdash}}[\Box_L^T]}{\Diamond\Box A, A^{\vdash}}[\Diamond_L]}{\Diamond\Box A \supset A^{\vdash}}[\supset_R]$$

- Redundancies :
  - $\Diamond\Box A, \langle \Box A, A, A\rangle, A^{\vdash} \lesssim \Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}$ et
  - $\Diamond\Box A, \langle \Box A\rangle, \langle \Box A, A\rangle, A^{\vdash} \lesssim \Diamond\Box A, \langle \Box A, A\rangle, A^{\vdash}$.

We deduce that $\Diamond\Box A \supset A^{\vdash}$ is not valid in IK$\{T\}$

Plan

A Multi-contextual Structure : MC-sequent

A MC-sequent is a multi-contextual structure of the form :

$$\Gamma_1; \ldots ; \Gamma_k \vdash \Gamma \vdash C$$

- $\forall i \in [1, k]$, Γ_i multi-sets of formulae (Contexts)
- Γ is a multi-set of formulae (Current context)
- C is a formula (Conclusion)
- Corresponding formula :

$$(\Diamond(\bigwedge \Gamma_1) \wedge \ldots \wedge \Diamond(\bigwedge \Gamma_k)) \supset ((\bigwedge \Gamma) \supset C)$$

- Spatial distribution of the assumptions
- MC-sequents are not hypersequents :

$$\Gamma_1 \vdash \mid \ldots \mid \Gamma_k \vdash \mid \Gamma \vdash C$$
$$\Downarrow$$
$$\Box(\bigwedge \Gamma_1 \supset \bot) \vee \ldots \vee \Box(\bigwedge \Gamma_k \supset \bot) \vee \Box(\bigwedge \Gamma \supset C)$$

A sequent calculus for IS5

Axioms and right rules of G_{IS5}

- Axioms :

$$\frac{}{G \vdash \Gamma, A \vdash A}\ [Id] \qquad \frac{}{G \vdash \Gamma, \bot \vdash C}\ [\bot^1] \qquad \frac{}{G; \Gamma', \bot \vdash \Gamma \vdash C}\ [\bot^2]$$

- Two cut rules :

$$\frac{G \vdash \Gamma \vdash A \qquad G \vdash \Gamma, A \vdash C}{G \vdash \Gamma \vdash C}\ [Cut^1] \qquad \frac{G; \Gamma \vdash \Gamma' \vdash A \qquad G; \Gamma', A \vdash \Gamma \vdash C}{G; \Gamma' \vdash \Gamma \vdash C}\ [Cut^2]$$

- Right rules :

$$\frac{G \vdash \Gamma \vdash A \qquad G \vdash \Gamma \vdash B}{G \vdash \Gamma \vdash A \wedge B}\ [\wedge_R] \qquad \frac{G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash A \vee B}\ [\vee_R^1]$$

$$\frac{G \vdash \Gamma \vdash B}{G \vdash \Gamma \vdash A \vee B}\ [\vee_R^2] \qquad \frac{G \vdash \Gamma, A \vdash B}{G \vdash \Gamma \vdash A \supset B}\ [\supset_R]$$

$$\frac{G; \Gamma \vdash \vdash A}{G \vdash \Gamma \vdash \Box A}\ [\Box_R] \qquad \frac{G \vdash \Gamma \vdash A}{G \vdash \Gamma \vdash \Diamond A}\ [\Diamond_R^1] \qquad \frac{G; \Gamma \vdash \Gamma' \vdash A}{G; \Gamma' \vdash \Gamma \vdash \Diamond A}\ [\Diamond_R^2]$$

A sequent calculus for IS5

Left rules of G_{IS5}

- Two kinds of left rules (L-rules and LL-rules) :

$$\frac{G \vdash \Gamma, A \supset B \vdash A \qquad G \vdash \Gamma, A \supset B, B \vdash C}{G \vdash \Gamma, A \supset B \vdash C}\ [\supset_L] \qquad \frac{G; A \vdash \Gamma, \Diamond A \vdash C}{G \vdash \Gamma, \Diamond A \vdash C}\ [\Diamond_L]$$

$$\frac{G \vdash \Gamma, \Box A, A \vdash C}{G \vdash \Gamma, \Box A \vdash C}\ [\Box_L^1] \qquad \frac{G; \Gamma', A \vdash \Gamma, \Box A \vdash C}{G; \Gamma' \vdash \Gamma, \Box A \vdash C}\ [\Box_L^2]$$

$$\frac{G; \Gamma \vdash \Gamma', A \supset B \vdash A \qquad G; \Gamma', A \supset B, B \vdash \Gamma \vdash C}{G; \Gamma', A \supset B \vdash \Gamma \vdash C}\ [\supset_{LL}]$$

$$\frac{G; \Gamma', \Box A \vdash \Gamma, A \vdash C}{G; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^1] \qquad \frac{G; \Gamma', \Box A, A \vdash \Gamma \vdash C}{G; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^{2a}]$$

$$\frac{G; \Gamma'', A; \Gamma', \Box A \vdash \Gamma \vdash C}{G; \Gamma''; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^{2b}] \qquad \frac{G; A; \Gamma', \Diamond A \vdash \Gamma \vdash C}{G; \Gamma', \Diamond A \vdash \Gamma \vdash C}\ [\Diamond_{LL}]$$

A sequent calculus for IS5

Properties of the G_{IS5} calculus :

- Soundness and completeness of G_{IS5} for IS5.
- Cut-elimination property and subformula property.
- New decision procedure for IS5
  - A preorder on MC-sequent
  - A notion of redundant derivation
  - Decision : search of irredundant proof of the MC-sequent
- New syntactic proof of decidability for IS5

Plan

Conclusion and perspectives

- A new structure : T-sequent
- Label-free sequent calculi for intuitionistic modal logics based on T, B, 4 et 5
- Cut-elimination property and subformula property
- Decision procedures in some cases

More details in **Journal of Logic and Computation, 2015**

- A new structure : MC-sequent
- Label-free sequent calculi for IS5
- Cut-elimination property and subformula property
- A decision procedure for IS5

More details in **LPAR proceedings, 2010**

Conclusion and perspectives

- New results of decidability (syntactic proofs)
  - IK$\{4\}$ and IS4 : another structure, another notion of redundancy
  - IK$\{5\}$, IK$\{4,5\}$ and IB4 : variants of T-sequent or MC-sequent.
- New decision procedures and improvements of existing ones
- Complexity of proof-search in intuitionistic modal logics
- Study of proof-theory in intermediate logics
- Combination of proof-search with countermodel generation in intutionistic modal logics.