Semantic based labelled sequent calculi for non-normal modal logics

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Outline

(1) Non-normal modal logics

(2) Labelled sequent calculi for basic non-normal modal logics

(3) Mapping: from internal to external calculi

(3) Open problems and future work

Non-normal modal logics

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Non-normal modal logics

Non-normal modal logics are obtained by adding to classical propositional logic the rule of inference

$$\text{(RE)} \quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

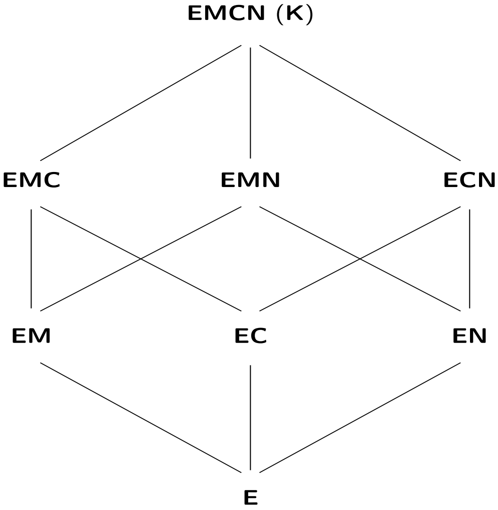
and any combination of the axioms

$$\begin{array}{ll} \text{(M)} & \Box(A \wedge B) \rightarrow \Box A \wedge \Box B \\ \text{(C)} & \Box A \wedge \Box B \rightarrow \Box(A \wedge B) \\ \text{(N)} & \Box \top \end{array}$$

Notation

The resulting systems are denoted with $\mathbf{ES}_1\ldots\mathbf{S}_n$, where $\mathrm{S}_i \in \{\mathrm{M,C,N}\}$.

The cube of non-normal modal logics



Neighbourhood semantics

Neighbourhood frames

A *neighbourhood frame* is a pair $\mathcal{F} = \langle W, I \rangle$ where

- W is a non empty set and
- I is a function $W \longrightarrow \mathcal{P}\mathcal{P}(W)$.

Neighbourhood models

A *neighbourhood model* is a triple $\mathcal{M} = \langle W, I, V \rangle$ where

- $\langle W, I \rangle$ is a neighbourhood frame and
- V is a valuation function for atomic formulas.

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- V is a valuation function for atomic formulas.

Truth in a world

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & w \in V(p); \\
\mathcal{M}, w \not\models \bot; & & \\
\mathcal{M}, w \models A \wedge B & \text{iff} & \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models A \rightarrow B & \text{iff} & \mathcal{M}, w \not\models A \text{ or } \mathcal{M}, w \models B; \\
\mathcal{M}, w \models \Box A & \text{iff} & \|A\| \in I(w), \text{ where } \|A\| = \{v \in W \mid \mathcal{M}, v \models A\}.
\end{array}
$$

Neighbourhood semantics

Frame properties

A neighbourhood frame $\mathcal{F} = \langle W, I \rangle$

- is *supplemented* if: if $\alpha \in I(w)$ and $\alpha \subseteq \beta$, then $\beta \in I(w)$;
- is *closed under intersections* if: if $\alpha, \beta \in I(w)$, then $\alpha \cap \beta \in I(w)$;
- *contains the unit* if: for all $w \in W$, $W \in I(w)$.

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Correspondence

- M is valid on $\mathcal{F}$ iff $\mathcal{F}$ is supplemented.
- C is valid on $\mathcal{F}$ iff $\mathcal{F}$ is closed under intersections.
- N is valid on $\mathcal{F}$ iff $\mathcal{F}$ contains the unit.

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- N is valid on $\mathcal{F}$ iff $\mathcal{F}$ contains the unit.

Completeness

E(**M**/**C**/**N**) is sound and complete w.r.t. the class of all the neighbourhood frames (that are supplemented/closed under intersections/contain the unit).

Sequent calculi for basic non-normal modal logics

Internal calculi

- Lavendhomme, R., and T. Lucas, *Sequent Calculi and Decision Procedures for Weak Modal Systems*, «Studia Logica», 65 (2000), pp. 121-145.
- Lellmann, B. and E. Pimentel, *Proof Search in Nested Sequent Calculi*, in M. Davis et al. (eds.), Proceedings of LPAR-20, 2015, pp. 558-574.

State of the art

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External calculi

- Gilbert, D. and P. Maffezioli, *Modular Sequent Calculi for Classical Moldal Logics*, «Studia Logica», 103 (2015), pp. 175-217.
- Negri, S., *Proof theory for non-normal modal logics: The neighbourhood formalism and basic results*, «IfCoLog», 4 (2017), pp. 1241-1286.

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Further internal calculi by Indrzejczak and Orlandelli for logics beyond the cube (see References).

Global aim

Desiderata

- Analyticity
- Standardness: finite numbers of rules each one with finite and fixed number of premisses.
  - Stronger requirement: each modal rule introduces exactly one modal formula.
- Modularity
- Termination of the decision procedure
- Optimal complexity
- Countermodel generation: obtain directly a countermodel from a failed proof search.

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Long term goal

- Give labelled calculi satisfying the desiderata for all logics of the cube.
- Condider extensions with axioms T, D, 4, 5 and B.

Labelled sequent calculus **G3E**

Negri (2017)

G3E Negri (2017)

Enriching the language

- Labels: world labels $x, y, z \dots$; neighbourhood labels: $a, b, c \dots$.
- Expressions: $a \in I(x)$, $x \in a$, $x : A$, $a \Vdash^{\forall} A$, $A \lhd a$.

G3E Negri (2017)

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Rules of **G3E**

Initial sequents $x : p, \Gamma \Rightarrow \Delta, x : p \qquad x \in a, \Gamma \Rightarrow \Delta, x \in a$

Propositional rules As for **G3K**

Rules for $\Box$

$$\frac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} \; L\Box \text{ (a fresh)}$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A \qquad a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd a}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A} \; R\Box$$

Rules of **G3E**

Rules for local forcing

$$\frac{x \in a, x : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L \Vdash^{\forall} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R \Vdash^{\forall} \text{ (}x\text{ fresh)}$$

Rules for $\lhd$

$$\frac{A \lhd a, \Gamma \Rightarrow \Delta, x : A \qquad x \in a, A \lhd a, \Gamma \Rightarrow \Delta}{A \lhd a, \Gamma \Rightarrow \Delta} \; L\lhd^{*}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta, x \in a}{\Gamma \Rightarrow \Delta, A \lhd a} \; R\lhd \text{ (}x\text{ fresh)}$$

* Cf. Correction note.

G3E Negri (2017)

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* Cf. Correction note.

Remarks

- Too much syntax, not justified by the semantics.
- $L\lhd$ is computationally expensive.

G3^{C}E

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Equivalent conditions for $\mathcal{M}, w \models \Box A$

(1) $\|A\| \in I(w)$.

(2) There is an $\alpha \in I(w)$ such that $\alpha \subseteq \|A\|$ and $\|A\| \subseteq \alpha$.

(3) There is an $\alpha \in I(w)$ such that $\alpha \subseteq \|A\|$ and $\alpha^c \cap \|A\| = \emptyset$.

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Idea

Rewrite the calculus by using (3).

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Rewrite the calculus by using (3).

Two kinds of neighbourhood labels

- positive neighbourhood labels $a, b, c, \ldots$
- negative neighbourhood labels $\overline{a}, \overline{b}, \overline{c}, \ldots$

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A new $L\Box$ rule

$$\frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, \overline{a} \Vdash^{\exists} A}{x : \Box A, \Gamma \Rightarrow \Delta} \; L\Box \; (a \text{ fresh})$$

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Initial sequents $x : p, \Gamma \Rightarrow \Delta, x : p$

Propositional rules As in Negri 2017

Rules for $\Box$

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Rules for local forcing

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G3^{C}E

Basic structural properties

- Weakening and contraction are height-preserving admissible;
- All the rules are height-preserving invertible;
- The cut rule is admissible (syntactic cut elimination).

$\mathbf{G3}^C\mathbf{E}$

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Completeness

$\mathbf{G3}^C\mathbf{E}$ is equivalent to $\mathbf{E}$:

$$\mathbf{E} \vdash A \quad \text{iff} \quad \mathbf{G3}^C\mathbf{E} \vdash \Rightarrow x : A \text{ for all } x.$$

Completeness

$\mathbf{G3}^C\mathbf{E}$ is sound and complete w.r.t. the class of all the neighbourhood frames.

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Completeness

G3^{C}E is sound and complete w.r.t. the class of all the neighbourhood frames.

Termination of the decision procedure

Proof search for $\Rightarrow x : A$ in **G3^{C}E** is terminating.

G3^{C}E

Admissible rule

A crucial issue for semantic completeness of the calculus is the admissibility of the rule

$$\frac{x \in a, \Gamma \Rightarrow \Delta \qquad x \in \overline{a}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Digression: Weak neighbourhood semantics

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Weak neighbourhood frames

A *weak neighbourhood frame* is a pair $\mathcal{F} = \langle W, I \rangle$ where

- W is a non empty set and
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Digression: Weak neighbourhood semantics

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$\mathbf{G3}^C\mathbf{E}$ is sound and complete w.r.t. the class of all the weak neighb. frames.

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Completeness

G3^C**E** is sound and complete w.r.t. the class of all the weak neighb. frames.

Completeness (corollary)

E is sound and complete w.r.t. the class of all the weak neighbourhood frames.

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Completeness (corollary)

E is sound and complete w.r.t. the class of all the weak neighbourhood frames.

Semantic equivalence (corollary)

$\models_{St} A$ iff $\models_{Wk} A$.

Extending $\mathbf{G3}^{C}\mathbf{E}$ with N, C and M

Extending $\mathbf{G3}^C\mathbf{E}$

(N) $W \in I(w)$ for all $w \in W$.

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Negri (2017)

$$\frac{a \in I(x), \top \lhd a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; N \; (a \text{ fresh})$$

where x is in $\Gamma \cup \Delta$.

Extending $\mathbf{G3}^C\mathbf{E}$

(N) $W \in I(w)$ for all $w \in W$.

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$$\frac{a \in I(x), \top \lhd a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; N \; (a \text{ fresh})$$

where x is in $\Gamma \cup \Delta$.

New rules

Add a neighbourhood constant τ, interpreted as W.

$$\frac{\tau \in I(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; N\tau$$

$$\frac{}{x \in \overline{\tau}, \Gamma \Rightarrow \Delta} \; N\overline{\tau}$$

where x is in $\Gamma \cup \Delta$ in rule $N\tau$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{y : \bot, y \in \tau, \tau \in I(x), \Rightarrow x : \Box\top, y : \bot}\; L\bot
    }{y \in \tau, \tau \in I(x), \Rightarrow x : \Box\top, y : \top}\; R\to
  }{\tau \in I(x), \Rightarrow x : \Box\top, \tau \Vdash^{\forall} \top}\; R\Vdash^{\forall}
  \qquad
  \dfrac{
    \dfrac{}{y \in \overline{\tau}, y : \top, \tau \in I(x), \Rightarrow x : \Box\top}\; N\overline{\tau}
  }{\overline{\tau} \Vdash^{\exists} \top, \tau \in I(x), \Rightarrow x : \Box\top}\; L\Vdash^{\exists}
}{
  \dfrac{\tau \in I(x), \Rightarrow x : \Box\top}{\Rightarrow x : \Box\top}\; N\tau
}\; R\Box
$$

Extending $\mathbf{G3}^C\mathbf{E}$

(C) If $\alpha, \beta \in I(w)$, then $\alpha \cap \beta \in I(w)$, for all $\alpha, \beta \subseteq W$.

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$$\frac{a \cap b \in I(x), a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ C$$

$$\frac{x \in a, x \in b, x \in a \cap b, \Gamma \Rightarrow \Delta}{x \in a \cap b, \Gamma \Rightarrow \Delta}\ L\cap$$

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Remark

Infinitely many terms: order and repetitions count.

Extending $\mathbf{G3}^C\mathbf{E}$

Two possible solutions

- Restrict the applicability of rule C in some opportune way.
- Define neighbourhood terms as sets.

Neighbourhood terms as sets

Positive neighbourhood terms are sets of positive neighbourhood labels and are written $(a_1, ..., a_n)$.

$$\frac{t_1 \cup t_2 \in I(x), t_1 \in I(x), t_2 \in I(x), \Gamma \Rightarrow \Delta}{t_1 \in I(x), t_2 \in I(x), \Gamma \Rightarrow \Delta} \; C$$

$$\frac{x \in t_1, x \in t_2, x \in t, \Gamma \Rightarrow \Delta}{x \in t, \Gamma \Rightarrow \Delta} \; L\cap' \text{ (if } t_1 \cup t_2 = t\text{)}$$

$$\frac{x \in \overline{t_1}, x \in \bar{t}, \Gamma \Rightarrow \Delta \qquad x \in \overline{t_2}, x \in \bar{t}, \Gamma \Rightarrow \Delta}{x \in \bar{t}, \Gamma \Rightarrow \Delta} \; L\overline{\cap} \text{ (if } t_1 \cup t_2 = t\text{)}$$

$$\dfrac{\dots, y \in \overline{a}, y : A, y : B \Rightarrow \overline{a} \Vdash^{\exists} A, y : A \dots}{\dots, y \in \overline{a}, y : A, y : B \Rightarrow \overline{a} \Vdash^{\exists} A \dots}\; R \Vdash^{\exists} \qquad \dfrac{\dots, y \in \overline{b}, y : A, y : B, \Rightarrow \overline{b} \Vdash^{\exists} B, y : B, \dots}{\dots, y \in \overline{b}, y : A, y : B, \Rightarrow \overline{b} \Vdash^{\exists} B, \dots}\; R \Vdash^{\exists}$$

$$\dots, y \in \overline{(a, b)}, y : A, y : B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, \dots \quad L\overline{\sqcap}$$

$$\dots, y \in \overline{(a, b)}, y : A \wedge B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, \dots \quad L\wedge$$

(*)

$$\dots, \overline{(a, b)} \Vdash^{\exists} A \wedge B \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B, \dots \quad L \Vdash^{\exists}$$

$$(a, b) \in I(x), a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow x : \Box(A \wedge B), \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B \quad R\Box$$

$$a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow x : \Box(A \wedge B), \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} B \quad C$$

$$x : \Box A, x : \Box B \Rightarrow x : \Box(A \wedge B) \quad L\Box(2)$$

$$\dfrac{\dots, y : A, y : B \Rightarrow y : B, \dots \qquad \dots, y : A, y : B \Rightarrow y : A, \dots}{\dots, y \in a, y \in b, y : A, y : B, y \in (a, b), a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, \dots}\; R\wedge$$

$$\dots, y \in a, y \in b, y \in (a, b), a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, \dots \quad L \Vdash^{\forall}(2)$$

$$\dots, y \in (a, b), a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow y : A \wedge B, \dots \quad L\cap'$$

(*) $\dots, a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Rightarrow (a, b) \Vdash^{\forall} A \wedge B, \dots \quad R \Vdash^{\forall}$

Extending $\mathbf{G3}^{C}\mathbf{E}$

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- M is not applicable in **G3E** (also if rules for $\subseteq$ are taken).

- In general, it seems that supplementation cannot be expressed by a "good" semantic rule.
  $w \models \Box(A \wedge B)$
  $\|A \wedge B\| \in I(w)$
  $\|A\| \in I(w)$
  Because $\|A \wedge B\| \subseteq \|A\|$: there is the truth-set $\|A\|$ and $\|A \wedge B\| \subseteq \|A\|$.
  This existential step is obvious (implicit) when reasoning semantically but it isn't obvious for the proof system (cf. Gilbert and Maffezioli (2015)).

Extending $\mathbf{G3}^C\mathbf{E}$

Negri (2017)

M is obtained by modifying (simplifying) rules for $\Box$ (a fresh in $L\Box m$).

$$\frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box m \qquad \frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\ R\Box m$$

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Our proposal

- In our calculus the same rules are obtained by removing existential forcings instead of $\lhd$-statements.
- For monotonicity we take the rules by Negri (2017).

Extending $\mathbf{G3}^C\mathbf{E}$

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Consequence

Calculi for monotonic and non-monotonic logics have separate but parallel lives: Extensions are obtained by adding the same rules (or sometimes a simplified versions for monotonic case).

Mapping: from internal to external calculi

Internal calculi Lavendhomme and Lucas (2000)

Eseq: $(CB)+(E)\ \dfrac{P\Rightarrow Q \qquad Q\Rightarrow P}{\Box P\Rightarrow \Box Q}$.

Mseq: $(CB)+(M)\ \dfrac{P\Rightarrow Q}{\Box P\Rightarrow \Box Q}$.

ECseq: $(CB)+(En)\ \dfrac{P_1,\dots,P_n\Rightarrow Q \qquad Q\Rightarrow P_1 \qquad \dots \qquad Q\Rightarrow P_n}{\Box P,\dots,\Box P_n\Rightarrow \Box Q}\ (n\geq 1)$.

MCseq: $(CB)+(Mn)\ \dfrac{P_1,\dots,P_n\Rightarrow Q}{\Box P,\dots,\Box P_n\Rightarrow \Box Q}\ (n\geq 1)$.

MNseq: $(CB)+(M)\ \dfrac{P\Rightarrow Q}{\Box P\Rightarrow \Box Q}\ +\ (N)\ \dfrac{\Rightarrow Q}{\Rightarrow \Box Q}$.

MCNseq: $(CB)+(Mn)\ \dfrac{P_1,\dots,P_n\Rightarrow Q}{\Box P,\dots,\Box P_n\Rightarrow \Box Q}\ (n\geq 0)$.

Translation

For every world label x

- if $\Gamma = A_1, ..., A_n$, then $\Gamma^{t_x} = x : A_1, ..., x : A_n$.
- $(\Gamma \Rightarrow \Delta)^{t_x} = \Gamma^{t_x} \Rightarrow \Delta^{t_x}$.

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Theorem

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in an internal calculus, then its translation $(\Gamma \Rightarrow \Delta)^{t_x}$ is derivable in the corresponding external calculus for every label x.

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Example: rule M_n $(n \geq 1)$ of **MCseq** is derivable in **G3**^{C}**EMC**.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{..., y : A_1, ..., y : A_n \Rightarrow y : B, ...}{y \in a_i, y \in (a_1, ..., a_n), (a_1, ..., a_n) \in I(x), a_i \in I(x), a_i \Vdash^{\forall} A_i \Rightarrow x : \Box B, y : B}\ L\Vdash^{\forall *}}{y \in (a_1, ..., a_n), (a_1, ..., a_n) \in I(x), a_i \in I(x), a_i \Vdash^{\forall} A_i \Rightarrow x : \Box B, y : B}\ L\cap^*}{(a_1, ..., a_n) \in I(x), a_i \in I(x), a_i \Vdash^{\forall} A_i \Rightarrow x : \Box B, (a_1, ..., a_n) \Vdash^{\forall} B}\ R\Vdash^{\forall}}{(a_1, ..., a_n) \in I(x), a_i \in I(x), a_i \Vdash^{\forall} A_i \Rightarrow x : \Box B}\ R\Box m}{a_i \in I(x), a_i \Vdash^{\forall} A_i \Rightarrow x : \Box B}\ C^*}{x : \Box A_1, ..., x : \Box A_n \Rightarrow x : \Box B}\ L\Box m^*
$$

Open problems and future work

Extensions with T, D, 4, 5, B

- In form of internal calculi they are considered in the works of Indrzejczak (see References).
  - All monotonic logics obtained by adding any combinations of axioms T, D, 4, 5, B have a cut free calculus.
  - Only some of the non monotonic logics obtained by adding combinations of T, D, 4, 5, B have a cut free calculus.
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Correspondence

(T)	$\Box A \to A$	if $\alpha \in I(w)$, then $w \in \alpha$.
(D)	$\Box A \to \neg\Box\neg A$	if $\alpha \in I(w)$, then $\alpha^c \notin I(w)$.
(4)	$\Box A \to \Box\Box A$	if $\alpha \in I(w)$, then $\{v : \alpha \in I(v)\} \in I(w)$.
(5)	$\neg\Box A \to \Box\neg\Box A$	if $\alpha \notin I(w)$, then $\{v : \alpha \notin I(v)\} \in I(w)$.
(B)	$A \to \Box\neg\Box\neg A$	if $w \in \alpha$, then $\{v : \alpha^c \notin I(v)\} \in I(w)$.

Future work

(T) If $\alpha \in I(w)$, then $w \in \alpha$.

$$\frac{a \in I(x), x \in a, \Gamma \Rightarrow \Delta}{a \in I(x), \Gamma \Rightarrow \Delta}\, T$$

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$$\dfrac{\dfrac{\dfrac{x : A, x \in a, a \in I(x), a \Vdash^{\forall} A, \Rightarrow x : A, \overline{a} \Vdash^{\exists} A}{x \in a, a \in I(x), a \Vdash^{\forall} A, \Rightarrow x : A, \overline{a} \Vdash^{\exists} A}\ L \Vdash^{\forall}}{a \in I(x), a \Vdash^{\forall} A, \Rightarrow x : A, \overline{a} \Vdash^{\exists} A}\ T}{x : \Box A \Rightarrow x : A}\ L\Box$$

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(D) If $\alpha \in I(w)$, then $\alpha^c \notin I(w)$.

$$\frac{a \in I(x), b \in I(x), y \in a, y \in b, \Gamma \Rightarrow \Delta \qquad a \in I(x), b \in I(x), y \in \overline{a}, y \in \overline{b}, \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta} \; D \; (y \text{ fr.})$$

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$$\dfrac{\dfrac{(1) \qquad (2)}{a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} \neg A \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} \neg A}\ D}{x : \Box A, x : \Box \neg A \Rightarrow}\ L\Box^{(2)}$$

$$\dfrac{\dfrac{..., y : A \Rightarrow y : A, ...}{y : A, y : \neg A, y \in \overline{a}, y \in \overline{b}, a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} \neg A \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} \neg A}\ L\neg}{(1)\ y \in a, y \in b, a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} \neg A \Rightarrow \overline{a} \Vdash^{\exists} A, \overline{b} \Vdash^{\exists} \neg A}\ L \Vdash^{\forall(2)}$$

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(Dm)

$$\frac{a \in I(x), b \in I(x), y \in a, y \in b, \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ Dm \ (y \text{ fresh})$$

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Open problems and future work

Future work

(1) Extensions with T, D, 4, 5, B.

(2) Show the opposite mapping: form external to internal systems.

(3) Intuitionistic non-normal modal logics.

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(1) Extensions with T, D, 4, 5, B.

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Thank you!

References I

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The Dependence Theorem (cf. Indrzejczak (2010))

CPL + T $\vdash$ D
CPL + T + 5 $\vdash$ B
CPL + D + 4 + B $\vdash$ T
E + T + 5 $\vdash$ 4
E + B + 4 + D $\vdash$ 5
E + B + 5 + T $\vdash$ 4
E + B + T $\vdash$ N